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2 Frege–Russell numbers

Analysis or explication?

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For both Gottlob Frege and Bertrand Russell, providing a philosophical account of the concept of number was a central goal, pursued along similar logicist lines. In the present paper, I want to focus on a particular aspect of their accounts: their definitions, or re-constructions, of the natural numbers as equivalence classes of equinumerous classes. In other words, I want to examine what is often called the 'Frege–Russell conception of the natural numbers' or, more briefly, the *Frege-Russell numbers*. My main concern will be to determine the precise sense in which this conception was, or could be, meant to constitute an *analysis*.[1] I will be mostly concerned with Frege's views on the matter; but Russell will come up along the way, for illustration and comparison, as will some recent neo-Fregean suggestions.

The structure of the paper is as follows. In the first section, I sketch Frege's general approach. Next, I differentiate several kinds, or modes, of analysis, as further background. In the third section, I zero in on the equivalence class construction, raising the question of why it might, from a Fregean point of view, be seen as 'the right' construction. thus as *an analysis in a strong sense*. In the fourth section, I provide a contrasting, more conventionalist view of the matter, often associated with the Carnapian notion of *explication*, and expressed in some remarks by Russell. I then discuss the motivation for the Frege–Russell numbers in more depth. In the sixth section, I introduce a neo-Fregean alternative, to be examined along similar lines. I conclude with a general observation concerning the significance of the kinds of arguments available in this connection.

1 Frege's general approach

In providing a philosophical account of the concept of number, especially with respect to the natural numbers, Frege had four general goals. First, he wanted to account for the mathematical theory of the natural numbers, i.e. provide conceptual foundations for 'pure arithmetic'. Second, he wanted to account for the main applications of the natural numbers, thus providing foundations for 'applied arithmetic' as well. Third, these two accounts were meant to form parts of an integrated, systematic approach, based only on a small number of core concepts and applicable beyond arithmetic. And fourth, the treatment of arithmetic was to proceed along logicist lines, in the sense of relying exclusively on logical resources.

The basic form of Frege's resulting proposal, first sketched in *Die Grundlagen der Arithmetik* (1884) and spelled out in more detail in *Grundgesetze der Arithmetik* (1893/1903), is well known. But let me remind the reader of some important ingredients. Frege's fourth goal, his logicism, was motivated in two ways: by his dissatisfaction with the views about the foundations of arithmetic he found in the literature of his time, including various Kantian, psychologistic, empiricist and formalist ideas; by an insight that suggested a close relationship between logic and arithmetic, namely recognition of their shared generality, i.e. of the fact that both are applicable, not just to everything observable or intuitable, but to everything thinkable. This two-fold motivation was reinforced by Frege's success in developing a new, more powerful logic, first presented in *Begriffsschrift* (1879). It was used right away to analyse a part of arithmetic hitherto assumed to depend on extra-logical foundations: the successor relation and, with it, the principle of mathematical induction.

With his new logic in place, including the initial successes in its application, Frege could turn to a more comprehensive investigation of both pure and applied arithmetic. On the applied side, he argued that all ascriptions of number should be analysed as statements about concepts. He also analysed the assignment of the same number to two concepts in terms of the existence of a 1–1 correspondence between them, or between the corresponding classes. On the pure side, Frege argued that an analysis of the natural numbers 0, 1, 2, ... reveals them to be objects, or the corresponding expressions to have the logical form of object names, in contrast to numerical functions and concepts, or their corresponding expressions. And given what he had already achieved in *Begriffsschrift*, all that had to be added, then, were definitions of the number 0, as a certain logical object, and of the successor function (from n to $n + 1$), as a function from logical objects to logical objects. Finally, if the latter could be formulated using only notions already employed, such as those of concept, class and 1–1 mappability, Frege's approach would have the systematic unity he sought.

Frege's particular definitions of the natural numbers 0, 1, 2, ..., against that background, amount to introducing the Frege–Russell numbers. That is to say, he constructed them in terms of a succession of classes of equinumerous classes (or at first, of equinumerous concepts). Before considering that construction in more detail, let me make some general observations about the considerations leading up to it. In summarizing the corresponding steps, I have already used the term 'analyse' several times. What kind, or kinds, of analysis are involved in them, and then also in the introduction of the Frege–Russell numbers? Following a recent suggestion by Michael Beaney, it will be helpful to make several distinctions in this connection.

2 Kinds, or modes, of analysis

Generally speaking, to analyse something means to work our way back to something more fundamental, something by means of which what we started with is accounted for. Probing further into what can be involved in it, i.e. in such 'working back' and 'accounting for', Beaney distinguishes between three kinds, or modes, of analysis: an 'interpretive', a 'regressive' and a 'resolutive' mode. As he notes, typically several of these modes are involved at once in a particular analysis.[2] Let me explain these distinctions further and give some examples, so as then to come back to Frege.

The *resolutive* mode, or the resolutive sense, of analysis is probably the most familiar. When we say that something – a material, a thing, a concept, a proposition, a truth, etc. – is subjected to analysis in this sense, what we mean is that its constitutive components, together with its underlying structure, are identified and made explicit. This often takes the form of decomposing a whole into parts, as exemplified by the chemical analysis of some material in terms of the basic elements it contains. Sometimes the resolution is not decompositional in such a narrow sense, but involves the identification of different kinds of underlying structure, e.g. in the function-argument analysis of the content of a sentence.

While chemistry provides the paradigm example of resolutive, or even decompositional, analysis, the clearest example for *regressive* analysis comes from mathematics. This kind of analysis, or this sense in which something is accounted for in terms of something more fundamental, involves the going back to basic premises, principles or causes. Thus, a truth of Euclidean geometry or Peano arithmetic can be analysed by deriving it from the corresponding fundamental axioms. A somewhat different example is the way in which a physical phenomenon is analysed by explaining its generation in terms of fundamental forces and causal processes.

Cases of resolutive and of regressive analyses usually involve another aspect, or another mode of analysis, as well: the *interpretive* mode. Both in a chemical and in a mathematical analysis, as just described, what happens is that something is interpreted, i.e. investigated and made sense of, against the background of a systematic, more general and often innovative framework – the periodic table of elements and the relevant axiomatic system. This framework provides the means for the resolution or regression, or for both. (Instead of an 'interpretative' mode of analysis, one could also talk about a 'translational' or 'transformative' mode.)

It is not hard to see that all three of these modes of analysis are involved in Frege's logicist project. His new logical system provides him with the systematic background and framework for analysing arithmetic truths; and analysis is here meant both in the resolutive and in the regressive sense. It is as interpreted within Frege's new logic that the structure of propositions of applied and of pure arithmetic as well as the nature of numbers appear in a new light; and it is as translated into that framework that Frege can attempt to derive all pure arithmetic truths from logical principles alone, both tasks that were hopeless within Aristotelian logic.

With our later discussion in mind, let me introduce one more distinction, also suggested by Beaney. It will apply to all three modes of analysis identified so far. Beginning with the third, an interpretive analysis may be meant in a weak sense, as providing merely a useful *rephrasal*; or it may be meant in a strong sense, as providing a substantive *reduction*. Considering resolutive and regressive analyses helps to clarify further what is involved. The aim of a resolutive analysis in the reductive sense is to identify ontologically basic elements and structures; the aim of a regressive analysis in the reductive sense is to identify epistemologically basic truths. In both, so far hidden, but fundamental, and metaphysically significant commitments are revealed. Analysis in the sense of mere rephrasal is not meant to have such metaphysical import. Its goal is to be useful in other ways, e.g. by helping to avoid misunderstandings and by opening up new avenues of inquiry.

3 Frege, platonism and reductive analysis

Returning to Frege, above we were led to his construction of the natural numbers as equivalence classes of classes. Specifically, he defines the number 0 as the class of all classes equinumerous to $\{x \mid x \neq x\}$; the number 1 becomes the class of all classes equinumerous to $\{x \mid x = 0\}$; the number 2, the class of all classes equinumerous to $\{x \mid x = 0 \text{ or } x = 1\}$; and so on. Without going into further detail about the 'and so on' (spelled out in terms of a logicized successor function and the notion of following in a series), I now want to raise my main question: what precisely is the status of these definitions? Using the distinctions introduced in the previous section: are Frege's definitions meant to involve one or several of our three modes of analysis: resolutive, regressive and interpretive? And are they meant as substantive reductions or as mere rephrasals?

We already noted the general interpretive dimension of Frege's project: the introduction of his new logic as the novel framework. To acknowledge that much is uncontroversial, I think. Controversies start when we ask: was this interpretative move meant to result in a substantive reduction or not? In the rest of this section, I want to consider a strong affirmative answer to this question. According to that answer, Frege's logicism has reductive significance along specific regressive and resolutive lines. On the regressive side, the claim is that arithmetic truths 'really' are based on the logical truths identified by Frege, in the sense that their derivations from them accords with the 'natural order' of truths. On the resolutive side, the claim is that the underlying structure of the natural numbers 'really' is that of the corresponding equivalence classes. The latter may also be expressed by saying that Frege's particular definitions, far from being mere stipulations, are meant to get at 'the fact of the matter'; they should be taken as 'correct' or 'the right ones', i.e. as veridical with respect to the nature of numbers.

What might lead a reader of Frege's works in this direction, towards saddling him with such strong views about analysis? This is not just a hypothetical question, as such a reading is probably the majority view, especially among his critics. Indeed, it seems not so far-fetched, at least on the surface. After all, doesn't one get the sense, from reading either *Grundlagen* or *Grundgesetze*, that Frege thought he had provided the definitive account of the foundations of arithmetic? And doesn't providing such an account require having hit upon 'the fact of the matter', including the 'real nature' of numbers? As such statements are still rather vague, let me consider two ways of spelling them out further. Both appeal to Frege's alleged platonism, in different ways.

Frege is often classified as an archetypical platonist with respect to mathematics. A first way of arriving at such a classification is by taking certain Fregean remarks from *Grundlagen* very seriously and, in a sense, literally, including the following: 'For number is no whit more an object of psychology or a product of mental process than, let us say, the North Sea is . . . It is something objective' (Frege 1884: 34); '[E]ven the mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there and give it a name' (ibid.: 107–8). Adding to these Frege's later comments, in the article 'The Thought', about a 'third realm' in which, presumably, numbers are to be located (Frege 1997: 337), how could he be interpreted as anything other than a platonist in the strongest possible sense?

Especially when taken out of context, Fregean remarks such as these conjure up the picture of a 'platonic heaven' of mathematical and other abstract objects, parallel to the spatio-temporal and empirically accessible universe, a realm by comparison with which our arithmetic statements are adjudicated. Elsewhere I have argued against such a reading of Frege.[3] But assuming for the moment that this is Frege's position, how does it lead to taking the Frege–Russell numbers as more than mere stipulations? Well, if it is a matter for the mathematician to 'discover what is there', this means, presumably, not just to discover that propositions such as $2 + 3 = 5$ are true, but also what the 'real nature' of the natural numbers is. And how do we discover the latter? Basically the same way in which a chemist discovers the constitution of water as H_20: we just look and see.

The problem is, of course, that it is not clear what such 'looking and seeing' amounts to in the present case. According to a widespread understanding of platonism, it involves a kind of quasi-perceptual access, parallel to the sense perception involved in the chemistry case – a sixth sense, as it were, or a 'mathematical intuition'. Apart from the general implausibility of such a view, this is problematic as a reading of Frege because he does not appeal to such a sixth sense anywhere in his writings. Moreover, even if one grants the possibility of a platonic sixth sense, there remain questions about how exactly it could reveal numbers to be composed in some specific way. After all, in chemistry it is also not a matter of mere 'looking and seeing'.

A more charitable and more sophisticated interpretation of Frege as a platonist has recently been presented by Tyler Burge.[4] His interpretation is not so much based on the quotations above, but on passages such as the following, also from *Grundlagen*: according to Frege, it took 'immense intellectual effort' to get clear about the concept of number; and this involved 'stripping off the irrelevant accretions which veil it from the eyes of the mind' (Frege 1884: vii). Or as Frege puts it in notes for 'Logic in Mathematics' (1914), it took a long time for us to arrive at a 'clear grasp' of the senses of certain signs, including numerical signs; since '[their] outlines [were for us] confused as if we saw [them] through a mist' (Frege 1979: 211).

According to Burge, Frege was after 'the fact of the matter' concerning the natural number. However, this is now not thought of in terms of some platonic sixth sense. Rather, it is a matter of reasoning and theory construction – so not of empiricist or quasi-empiricist observation, but of rationalist inquiry. And what exactly is supposed to be the result of such inquiry? We are, to use Frege's words again, 'getting clear about the concept of number'; or we are grasping the corresponding 'senses of numerical signs'. In Burge's paraphrase, we are acquiring a full understanding of those concepts or senses. Once such understanding is achieved, both Frege's basic logical laws and his logicist definitions are meant to become 'self-evident'.[5]

What happens in Burge's reading of Frege is that the attribution of a 'robust platonism' is combined with the attribution of a sophisticated rationalist epistemology to him. The former amounts to the view that what successful rationalist inquiry reveals are conceptual facts that are independent of us as inquirers; or again, what we come to understand are the relevant senses of signs as something existing 'out there', determinate in itself. At this point, the question arises again how such inquiry can possibly lead to the particular definitions of the natural numbers proposed by Frege. Burge doesn't offer much in that connection; he focuses on the general outlines of the position and its application to logical laws. I will attempt to fill that gap in later sections of this paper.

4 Russell, convention and Carnapian explication

According to both the simple platonism attributed to Frege by his critics and Burge's sophisticated reading, Frege's particular definitions of the natural numbers are more than mere stipulations – they are meant to correctly reflect the 'real nature' of numbers or the 'fully understood' senses of numerical signs. However, this is not a generally accepted position in the secondary literature. Even among readers who take Frege to be a platonist in a strong sense, some, like Michael Dummett, see these definitions as the point where a 'conventionalist strain' enters his views (Dummett 1991: 177). Understood as such, the definitions are only justified, or justifiable, in a weaker sense. What matters is simply that they allow for a systematic reconstruction of arithmetic on logical grounds. Any such reconstruction

will do, within certain general constraints. In that sense, there is no 'uniquely correct' one, much less a 'fact of the matter' that needs to be reflected.[6]

If we turn away from Frege briefly, it is essentially such a position that can be found in some of Bertrand Russell's writings, connected with his own introduction of the Frege-Russell numbers. As he writes in *Introduction to Mathematical Philosophy*:

> So far we have not suggested anything in the slightest degree paradoxical. But when we come to the actual definitions of the numbers we cannot avoid what must at first sight seem a paradox, though this perception will soon wear off. We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples. It is indubitable and not difficult to define, whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematic number 2, which must always remain elusive. Accordingly we set up the following definitions: . . . At the expense of a little oddity, this definition secures definiteness and indubitability; and it is not difficult to prove that numbers so defined have all the properties that we expect numbers to have.
>
> (Russell 1919: 14)

Note Russell's denial that he has captured, or even aimed at capturing, the 'real nature' of the number 2; he has no interest in getting hold of 'a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down'. What matters, instead, is 'definiteness and indubitability', as well as having an approach that allows us 'to prove that numbers so defined have all the properties that we expect numbers to have'.[7]

Attributing to either Frege or Russell a 'conventionalist strain' in this connection means – in terms of our earlier distinctions – that the Frege–Russell numbers are not meant as a resolutive analysis in the reductive sense. But this raises additional questions: if we deny a reductive sense to this part of their proposals, what follows for the rest, including the regressive and interpretive parts; do we have to conclude that they, too, cannot be meant in a reductive sense? And if so, isn't that in tension with the usual understanding of their goals; isn't it usually assumed that logicism involves a claim about what arithmetic truths are ultimately based on, and isn't it assumed that it involves getting the overall logical framework right? It seems that, if we take the Frege–Russell numbers to be mere conventions, with no deeper claim at veridicality, there is pressure to take back such claims as well, since they are interrelated.

A philosopher who takes these issues head on is Rudolf Carnap. According to Carnap, the Frege–Russell numbers and similar constructions should be seen as results of applying the method of *explication*. As he writes in *Meaning and Necessity* (1947):

> The task of making more exact a vague or not quite exact concept used in everyday life or in an earlier stage of scientific or logical development, or rather of replacing it by a newly constructed, more exact concept, belongs among the most important tasks of logical analysis and logical construction. We call this the task of explicating, or of giving an *explication* for, the earlier concept; this earlier concept, or sometimes the term used for it, is called the *explicandum*; and the new concept, or its term, is called an *explicatum* of the old one. Thus, for instance, Frege and, later, Russell took as explicandum the term 'two' in the not quite exact meaning in which it is used in everyday life and in applied mathematics; they proposed as an explicatum for it an exactly defined concept, namely the class of pair-classes.
>
> (Carnap 1947: 7–8).[8]

Note again that, from Carnap's perspective, Frege's and Russell's logicist definitions of numbers are not meant to capture the earlier, ordinary meaning of terms such as 'two', since that meaning is, as Russell already remarked, inexact and hard to pin down. Rather, the new *explicatum* is meant to simply replace the old, vague *explicandum*.

Elsewhere, Carnap is also quite clear about two related points: the replacement of an *explicandum* by a corresponding *explicatum* usually involves providing a novel framework for talking about the latter; and the choice of both the framework and particular constructions within it can always only be justified pragmatically, in terms of how well they allows us to do what we want to do, not in some deeper metaphysical sense. In fact, already in Carnap's *The Logical Syntax of Language* the question of what the 'real nature' of the natural numbers is, including whether or not the Frege–Russell definitions capture it, is presented as a prime example of a metaphysical pseudo-problem, thus as something to be overcome (Carnap 1934/37: 300ff.). Explication in Carnap's sense is clearly not reductive analysis, in none of our three senses or modes.

Returning to Frege again, let me add one related observation. There is evidence that Carnap's notion of explication actually has roots in Frege's views, more precisely in what Carnap learned in a class taught by Frege at the University of Jena in 1914. In that class – 'Logic in Mathematics' – Frege expressed the following idea:

> If we have managed in this way [by giving a definition for a new sign B] to construct a system for mathematics without any need for the [old] sign A, we can leave the matter there; there is no need at all to answer the question concerning the sense in which – whatever it may be – this sign had been used earlier ... We must therefore explain that the sense in which this sign was used before the new system was constructed is no longer of any concern for us.
>
> (Frege 1979: 227–8)[9]

If we take the sign A to be 'two', as used in ordinary language, and the sign B to be 'the class of all pair-classes', as part of a logicist system, we are very close to Carnapian explication, aren't we? In a recent paper, Michael Beaney has argued for this claim.[10] Without going into any further detail, it is striking that his case is based on passages from the same texts used by Burge to support his interpretation. The two are led in diametrically opposed directions: Burge towards a reading of Frege as providing a reductive analysis; Beaney towards a reading of him as providing an explication.

5 Motivating the Frege–Russell construction further

We have arrived at some stark disagreements on how to interpret Frege, in general and concerning the status of the Frege–Russell numbers in particular. I will not attempt a final adjudication of these disagreements, as this would require a close, sustained textual analysis for which there is no room. Instead, I want to address a more systematic issue. But before doing that, let me at least make three brief observations concerning Frege's writings. They all illustrate that the textual evidence is not as easy to assess as one might think.

First, consider a notorious footnote in *Grundlagen* in which Frege comments on his use of the phrase 'extension of the concept' (used interchangeably with 'class determined by the concept', at least in Frege's later writings). He writes: 'I believe that for "extension of the concept" we could write simply "concept"' (Frege 1884: 80). As this remark occurs in direct connection with his definition of the natural numbers, the question arises: does it mean that using 'concept' instead of 'extension of the concept' would simply be a notational variant, i.e. leave the definition itself unchanged; or does it mean that Frege is allowing for an alternative definition, not using classes, which would work as well for his purposes?[11]

Second, note that the definitions of the natural numbers given in *Grundlagen* and in the later *Grundgesetze* are not identical. In the former, Frege uses equivalence classes of equinumerous concepts; it is only in the latter that he switches over to equivalence classes of (corresponding) equinumerous classes. Does this switch constitute an implicit acknowledgment by Frege, now in a different way, that two alternative constructions are possible; or does he hold that the first construction is inferior to the second in some strong sense? And in the case of the latter, is it because the new definition gets more at 'the fact of the matter' or, rather, because of pragmatic advantages?

Answers to these two questions are not obvious. In connection with the second, Frege's reasons for replacing the *Grundlagen* construction with that from *Grundgesetze* are crucial; but he is not very explicit about those reasons. The first question leads to thorny issues about the reference of terms such as 'the concept F'. The matter gets even murkier if we add a third question: do Frege's seemingly Carnapian remarks in the relatively late notes for 'Logic in Mathematics' represent a stable view; or do they, instead, indicate a development or even a radical change in his position? If the latter (as Beaney argues), this complicates the interpretive task further.

I want to move on. Assume for the moment that Frege takes his definition of the natural numbers as equivalence classes of equinumerous classes to be 'the right one', or at least to be privileged in some strong sense. The question I want to address is this: how could he, or anyone, possible argue for such a claim? Let me put aside right away the naïve platonist idea that we simply 'look and see', because saying so does not really help. Let us examine instead, along Burgean lines, which conceptual and theoretical considerations one could appeal to. Or put slightly differently, are there any such considerations that, taken as a whole, provide enough constraints on Frege's project that they single out the Frege–Russell numbers?

Reformulating the question as indicated leads to a further question: what are the goals of Frege's project? And this leads back to my brief sketch of that project above. One of the aims I attributed to Frege was that of providing a new foundation for pure arithmetic. In order to provide such a foundation, we need a system in which various basic arithmetic principles are provable. The most familiar set of such principles consists, of course, of the Dedekind–Peano Axioms. As Dedekind, Peano and others have taught us, these provide an axiomatic basis for arithmetic that is complete in the sense of categorical (and the latter is the best we can hope for, as follows from Gödel's results).[12] Moreover, Frege knew of a variant of this axiomatic basis and, at least to some degree, its categoricity.[13]

Taking such results into account, an initial constraint on any definition of the natural numbers is this: the constructed sequence of objects needs to satisfy the Dedekind–Peano Axioms, or any equivalent system such as Frege's. Put in metalogical language, it needs to form a model of the axioms. Frege had reason to believe that the Frege–Russell numbers do that. (I am putting aside the consistency question here; more on it later.) But any other model of the axioms will do so as well, as is often emphasized today. Thus, our first constraint is not strong enough to single out Frege's construction as the one and only possible one, much less as 'the right one'.

While any model of the Dedekind–Peano Axioms will, indeed, do for inner-mathematical purposes, this was not Frege's only concern. Recall that he had three additional goals, as parts of his broader, more ambitious project: to account for the applications of the natural numbers; to account for pure and applied mathematics in an integrated, systematic way; and to provide these accounts using logicist means alone. This leads to a new question: do these three goals provide enough additional constraints to narrow the choice of a model down further, perhaps even to a single one? Let us see how one could possibly make a case for the latter.

As Frege wants to account not just for pure arithmetic, but also its applications, what does that add? Two points: applied statements of number are statements about concepts; the same number is assigned to two concepts, or corresponding classes, if the objects falling under them can be correlated 1–1. But what does that imply; or, more suggestively, what are the numbers themselves, then? Well, they are ways of correlating concepts, or classes, according to their size (cardinality).[14] Russell puts the same point a bit more strongly: '[I]t is clear that number is a way of bringing together certain collections, namely those that have a given number of terms' (Russell 1919: 14). This leads to the following additional constraint: if we want to construct the natural numbers as cardinal numbers, like Frege, we need to build the idea of correlating, or even of 'bringing together', equinumerous classes into their very nature.[15]

There is more. Frege's analysis of the logical role of numerical expressions led him to the conclusion that numbers need to be seen as objects, not concepts. This rules out, among others, identifying them with second-order numerical concepts, which would have satisfied the previous constraint.[16] Frege also wants to provide a unified account of pure and applied mathematics; and he wants to do so in a logicist way. Consequently, numbers have to be constructed by logicist means alone; and it has to be done in a manner that integrates the satisfaction of all our constraints. Now, logic – as conceived of by Frege and Russell – provides us with a theory of classes, which count as objects. More specifically, it allows (or seems to allow) for the formation of equivalence classes of equinumerous classes, thus for capturing Russell's 'bringing together' very directly.

What the line of thought just rehearsed does, I would say, is to provide a very strong motivation for the Frege–Russell construction. But is it strong enough to single out one and only one construction, perhaps even as 'the right one'? Here is a way to resist such conclusions: assume we keep working within a logicist system. How about constructing, not the Frege–Russell numbers, but the *von Neumann numbers* (finite von Neumann ordinals) within this framework? That is to say, let us define 0 as $\emptyset$ (the empty or null set), 1 as $\{0\}$, 2 as $\{0, 1\}$, etc. These are certainly available as classes; and together they form a model of the Dedekind–Peano Axioms. But then, doesn't this provide an alternative to the Frege–Russell numbers?

To counter this argument, one might respond as follows: yes, the von Neumann numbers suffice for inner-mathematical purposes; and yes, they can be constructed, not just in contemporary set theory, but also within a logicist system. Yet they fail to satisfy Frege's other constraints. They do not have the required correlating of equinumerous classes built right into them, especially not in the form of a 'bringing together'; and as a consequence, the approach lacks the unity and systematicity that recommends the Frege–Russell numbers. Then again, aren't the von Neumann numbers still related to all the relevant classes in a rather direct way, namely by being equinumerous to them? And isn't equinumerosity absolutely central to Frege's approach, so that one core concept integrates the various parts of this approach?[17]

At this point, the debate turns on whether or not the von Neumann way of correlating all the right classes, while quite direct and formulated in terms of a central Fregean notion, captures the cardinal application of numbers 'as well as' the equivalence class construction. It is not clear to me how to address such a question, vague as it is.[18] Perhaps this is reason enough to deny that the Frege–Russell numbers have a privileged status, even taking into account all of Frege's constraints? Actually, let me add a further consideration, one that will undercut the present dichotomy.

6 Consistency and a neo-Fregean alternative

Some readers may have the following response to our discussion so far: what is the point of considering the deeper motivation for the Frege–Russell numbers, or possible arguments for their privileged status; after all, doesn't the inconsistency in Frege's logic undermine the whole approach? In other words, the Frege–Russell numbers don't have systematic interest any more; at best, they have minor historical interest.

Such a dismissive response is too quick; it overlooks that it might be possible to fix Frege's logic so as to allow for the resurrection of the Frege–Russell numbers. Here I do not have in mind Whitehead and Russell's *Principia Mathematica*, because the original equivalence class construction is not preserved in it (but splintered up into infinitely many levels and deformed in other ways as well). Rather, I would point in two other directions. First, consider W.V.O. Quine's *New Foundation*, a system in which the Frege–Russell construction can be repeated in its original form. This system is known to be consistent (relative to set theory), at least in the form of NF with urelements, as needed if we want to provide a framework, not just for pure, but also for applied mathematics. Second, there are some recent results by George Boolos to the effect that the introduction of the Frege–Russell numbers in itself does not lead to contradiction, although its combination with standard class- or set-theoretic principles does.[19]

I do not mean to suggest that we have, at this point, a workable and attractive logicist system available, one in which both the von Neumann and the original Frege–Russell numbers can be constructed. There are various well-known problems with Quine's NF; and Boolos' results certainly don't go that far. But what I mentioned indicates that it is not clear we will never be able to come up with such a system in the future, and perhaps that is enough for present purposes. Beyond that, there is other recent work motivated by concerns for consistency that is relevant for our purposes: the neo-Fregean or neo-logicist investigations by Crispin Wright, Bob Hale and others.[20]

Wright's and Hale's neo-logicist programme has led to an intense, complex debate, including various technical investigations. I will focus on just one aspect, of a more conceptual and metaphysical nature, and concerning their treatment of the natural numbers. The original insight in, and a main motivation for, that treatment is the following: if we look carefully at Frege's works, it becomes apparent that the introduction of the Frege–Russell numbers plays one and only one role, namely to allow for a derivation of the Cantor–Hume Principle, 'For all F and G, the number of Fs = the number of Gs if and only if F and G can be correlated 1–1.' Everything else, including Frege's version of the Dedekind–Peano Axioms, is derived from this principle within second-order logic. Moreover, and beyond Frege now, the system consisting of second-order logic and the Cantor–Hume principle is consistent (relative to set theory, indeed relative to second-order arithmetic). The suggestion is this, then: why not work with the latter directly, as it also gets us around the inconsistency in Frege's original system?

Wright and Hale emphasize that their procedure allows us to achieve all of Frege's goals; or at least it does so if we broaden our view of logicism slightly. (More on the latter in a moment.) Two aspects are especially noteworthy. On the one hand, their neo-logicist approach involves giving up the Frege–Russell numbers and, instead, treating expressions of the form 'the number of Fs' as primitive. Metaphysically speaking, numbers become basic objects; they are not constructed as classes any more, nor are they seen as composed of parts in any way.[21] On the other hand, precisely because of their introduction as 'the number of Fs', for various concepts F, the natural numbers are still tightly bound to their cardinal applications.

This adds to our earlier discussion as follows: the *Wright–Hale numbers* – as I will call them – seem to provide another alternative to the Frege–Russell numbers. All our constraints are satisfied: we account for pure arithmetic; we account for applied arithmetic; we do so in an integrated, systematic form; and we use (presumably) only logicist means. To be sure, the way in which the application of arithmetic is incorporated is slightly different now. But the correlating of all relevant concepts or classes remains central to the very definitions of the numbers; and it is integrated well with how the other constraints are satisfied. Perhaps this shows, once again, that the Frege–Russell numbers are not so privileged after all.

Actually, there is another possible reaction here, as one may want to push such considerations towards a different conclusion. To do so, two additional points need to be added. First, consider again the passage from Russell's *Introduction to Mathematical Philosophy* quoted at length above. It starts as follows:

> So far we have not suggested anything in the slightest degree paradoxical. But when we come to the actual definitions of the numbers we cannot avoid what must at first sight seem a paradox, though this perception will soon wear off. We naturally think that the class of couples (for example) is something different from the number 2.
>
> (Russell 1919: 14)

Note now 'what at first sight [seems] a paradox', namely that along Frege–Russell lines numbers are identified with certain classes, which seems odd. Russell tries to take the sting out of this oddity by remarking, right away, that any sense of it 'will soon wear off'. But is that really convincing – especially if we have the Wright–Hale numbers as an alternative? The challenge is this: if at all possible, shouldn't numbers be introduced *as numbers*, and not as something else (classes, etc.)?[22] And if so, doesn't that privilege the Wright–Hale numbers over both the Frege–Russell and the von Neumann numbers?

A second point adds to this challenge. Consider again the goal of constructing the natural numbers as cardinal numbers, by building the correlating of all relevant concepts or classes into their very nature. For the Frege–Russell numbers, this is achieved by forming corresponding equivalence classes; for the von Neumann numbers, by using classes of the right cardinality. In other words, in the first case we use the elementhood relation to do the correlating; in the second, the relation of being 1–1 mappable. But aren't both ways somewhat indirect and loaded with unnecessary structure, thus less than fully satisfying – especially when compared to the Wright–Hale numbers? What does the correlating in the latter case is a direct functional relation, and nothing more. Once again, doesn't that show it is the Wright–Hale numbers that are privileged?

I can see two ways of resisting such a singling out of the Wright–Hale numbers. First, perhaps the two points just presented are too weak, since based on notions that are hard to make precise and of dubious mathematical relevance. Second, in the end we have to face up to an issue postponed so far: does the Wright–Hale approach really work with a framework that should be classified as logicist? In particular, is the primitive nature of numbers, as assumed in it, really acceptable – are such objects really logical objects, if they are acceptable objects at all? And what about the status of the Cantor–Hume principle – is it really a logical principle, if it is acceptable as a basic principle at all? Both questions have certainly led to a lot of debate in the literature.[23]

7 A concluding observation

My main goal in this paper was not to argue that the Frege–Russell numbers are privileged in some strong sense, or even that this definition of the natural numbers is 'the right one'. Nor was it to establish that this holds, instead, for the Wright–Hale numbers. Two related goals were prior, and all I could pursue here. First, I wanted to clarify what is at issue in claiming, or denying, such a thing in the first place. I did so by distinguishing several kinds of analysis, including the contrast between reductive analysis, with strong metaphysical implications, and explication, conceived of in a more pragmatic way. Second, I wanted to explore what kinds of considerations could be adduced in a corresponding debate at all. Let me close with a general observation concerning both issues.

After quickly putting aside the idea of a naïve platonist 'look and see', what we explored were theoretical and conceptual considerations, the kind of considerations Tyler Burge could accept within his sophisticated platonist reading of Frege. In doing so, we were guided by the various goals underlying Frege's project. However, putting the matter in terms of goals and their satisfaction reveals the following: while Burge's platonism and Carnap's anti-metaphysical viewpoint are in stark contrast in general, a Carnapian can very well find room for such considerations. Instead of seeing them as revealing the 'deeper sense' of our pre-systematic signs, or the 'real nature' of the corresponding objects, they become part of the pragmatic dimension of explication. After all, any *explicatum* has to be evaluated for whether it allows us to achieve our goals or not.

If so, then the only significant difference between these two perspectives appears to be this: the Burgean presupposes that the kind of considerations discussed must, at least in principle, lead to a uniquely privileged definition in the end; otherwise we still don't understand the corresponding concepts or senses fully. For the Carnapian, it is an open question of whether there is a privileged definition or not. Perhaps it is even to be expected, from this point of view, that we will end up with various equally, or almost equally, useful alternatives. In any case, no intrinsic need is felt to find a best alternative, as several of them may allow us to reach our goals and as the search for a deeper 'fact of the matter' is explicitly abandoned.[24]

Notes

1 The paper builds on Reck 1997, 2000/2005a, 2003a and 2005b.
2 See Beaney 2000 and 2002.
3 See Reck 2000/2005a, also Reck 1997; both were influenced strongly by Ricketts 1986.
4 See the parts on Frege's 'rationalism' in Burge 2005, especially 'Frege on Knowing the Foundations'.
5 In Jeshion 2001, the role of self-evidence for Frege – concerning fundamental logical laws but also, presumably, his central logicist definitions – is emphasized even more.
6 Besides Dummett 1991, see Wilson 1992 and Demopoulos 1998.
7 As Russell's views are often a moving target, one may wonder whether this is a relatively late stance; but compare Russell 1903: 115–16, where a similar position is already taken.
8 For further elaborations concerning the notion of explication, see Carnap 1950: ch. 1.
9 In Carnap's own notes from this class similar passages occur; see Frege 2004: 140.
10 Beaney 2004.
11 Compare the following remark from later in *Grundlagen*: 'I attach no decisive importance even to bringing in the extensions of concepts at all' (Frege 1884: 117). Note, in addition, Frege's reference to 'fruitfulness' as what counts for definitions (ibid.: §§69–70).
12 See Awodey and Reck (2002) for a related historical and philosophical discussion.
13 See the discussion of some little-known passages from *Grundgesetze* in Heck 1993.

14 Here the use of the natural numbers as cardinal numbers is made central. Focusing instead on their ordinal use leads in a different direction; see Dedekind's views as discussed in Reck 2003b.
15 See Dummett 1991: ch. 20; compare also the discussion of 'Frege's Constraint' in Wright 2000.
16 For more on these second-order concepts, see Reck 2003a and 2005b.
17 The last three paragraphs summarize a debate between Michael Dummett and W.W. Tait; see Dummett 1991: ch.5 and the response, or corresponding challenge, in Tait 1997: 228–30.
18 A vague intuition may be that only the Frege–Russell numbers, but not the von Neumann numbers, are composed of all the relevant classes. But this misrepresents Frege's logical conception of class (or, more generally, of value range), developed in direct opposition to mereological ideas. Compare note 21.
19 See Reck 2005b, also for references. For the result that NF with urelements is relatively consistent, see Jensen 1969. I am grateful to Jamie Tappenden for pointing out to me that, for present purposes, such a version of NF is the relevant one.
20 See Hale and Wright 2001, as well as the references in it.
21 One may think that we have, thus, replaced a decompositional by a mere resolutive analysis. However, neither the Frege–Russell, nor the von Neumann, nor the Wright–Hale numbers exemplify decompositional analysis; all are cases of resolutive (function-argument) analysis more generally. Compare note 18.
22 The basic idea is that classes, of any kind, seem to have inappropriate (non-numerical) properties, such as having elements. Compare Dedekind's views as discussed in Reck 2003b.
23 A third question is how far, and in what precise way, the Wright–Hale approach can be generalized. As my focus in the present paper is on the natural numbers, I have put this important issue aside throughout.
24 I would like to thank my audience at the conference 'The Varieties of Analysis', St Catherine's College, Oxford, March 2005, for feedback on a first version of this paper, Michael Beaney and Teri Merrick for comments on later drafts, and Bill Demopoulos for helpful exchanges on related topics.

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