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FREGE'S NATURAL NUMBERS

Motivations and modifications

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Frege's main contributions to logic and the philosophy of mathematics are, on the one hand, his introduction of modern relational and quantificational logic and, on the other, his analysis of the concept of number. My focus in this paper will be on the latter, although the two are closely related, of course, in ways that will also play a role. More specifically, I will discuss Frege's logicist reconceptualization of the natural numbers with the goal of clarifying two aspects: the motivations for its core ideas; the step-by-step development of these ideas, from *Begriffsschrift* through *Die Grundlagen der Arithmetik* and *Grundgesetze der Arithmetik* to Frege's very last writings, indeed even beyond those, to a number of recent "neo-Fregean" proposals for how to update them.

One main development, or break, in Frege's views occurred after he was informed of Russell's antinomy. His attempt to come to terms with this antinomy has found some attention in the literature already. It has seldom been analyzed in connection with earlier changes in his views, however, partly because those changes themselves have been largely ignored. Nor has it been discussed much in connection with Frege's basic motivations, as formed in reaction to earlier positions. Doing both in this paper will not only shed new light on his response to Russell's antinomy, but also on other aspects of his views. In addition, it will provide us with a framework for comparing recent updates of these views, thus for assessing the remaining attraction of Frege's general approach.

I will proceed as follows: In the first part of the paper (§1.1 and §1.2), I will consider the relationship of Frege's conception of the natural numbers to earlier conceptions, in particular to what I will call the "pluralities conception", thus bringing into sharper focus his core ideas and their motivations. In the next part (§2.1 and §2.2), I will trace the order in which these ideas come up in Frege's writings, as well as the ways in which his position gets modified along the way, both before and after Russell's

antinomy. In the last part (§3.1 and §3.2), I will turn towards several more recent modifications of Frege's approach, paying special attention to the respects in which they are or aren't "Fregean". Along the way, I will raise various questions concerning the possibility of a Fregean approach to logic and the philosophy of mathematics.[1]

1.1 Frege's foil: The pluralities conception

From a contemporary point of view, as informed by nineteenth- and twentieth-century mathematics, it is perhaps hard to see the full motivation for, and the remaining attraction of, Frege's conception of the natural numbers. Not only are we keenly aware of the fact that Russell's antinomy undermines that conception (at least in its original form), but contributions by Dedekind, Peano, Hilbert, Zermelo, von Neumann, and others have also pushed us in the direction of adopting a formal-axiomatic, set-theoretic, or structuralist approach to the natural numbers. That is to say, either we start with the Dedekind-Peano Axioms and simply derive theorems from them, putting aside all questions about the nature of the natural numbers; or we identify these numbers with certain sets in the ZFC hierarchy, typically the finite von Neumann ordinals; or again, we take the natural number sequence to be the abstract structure exemplified by such set-theoretic models. It will be helpful, then, to examine in some detail what led Frege to his conception, or to the "Frege-Russell conception" as it is also often called. What, more particularly, were the main alternatives available at his time, and why did he replace them with his own?

The work in which Frege discusses alternative views about the natural numbers most extensively is *Die Grundlagen der Arithmetik* (Frege 1884). In that book, as well as some related polemical articles, Frege frequently presents himself as opposed to crude formalist and psychologistic positions. These are positions that identify the natural numbers with concrete numerals, on the one hand, or with images or ideas in the mind, on the other. Along the way, another opposition comes up as well, however, one that is more relevant for present purposes. Namely, Frege also reacts against the conception of numbers as "pluralities", "multitudes" or "groups of things". That general conception has a long history, from Mill and Weierstrass in the nineteenth century back to Aristotle, Euclid, and Ancient Greek thought. The basic idea behind it is this: Consider an equation such as "2 + 3 = 5". What is it we do when we use such a proposition? We assert that, whenever we have a plurality of two things and combine it with, or add to it, a plurality of three (different) things, we get a plurality of five things.

There are many questions one can raise about such a view, starting with what is meant by "plurality", by "combining" or "adding", and even by "thing". Traditional answers to those questions are not uniform, leading to a number of variants of the conception in question. But three basic, related aspects are shared by most of them:[2] First, even a simple arithmetic statement such as "2 + 3 = 5" is taken to be a universal statement ("whenever we have a plurality . . .", i.e., "for all pluralities . . ."). Second, a numerical term such as "2" (also "3", "2 + 3", etc.) is understood not as a singular term, referring to a particular object, but as a "common name" ("2" refers to all "couples", "3" to all "triples", etc.). And third, what we "name" along these lines are always several things considered together in some way, e.g., heaps of stones, flocks of sheep, or companies of soldiers ("pluralities", "multitudes" or "groups" in that sense).

So as to have a concise way of talking about it, let me call this general conception the "pluralities conception" of the natural numbers. It should be clear, even from the rough sketch just given, that it amounts to an essentially applied conception of arithmetic. It makes central the use of numbers in determining the "size" (cardinality) of groups of things. Indeed, numbers are simply identified with such groups, i.e., with "numbers of things" (with the result that there are many different "2s", "3s", etc.). Also clearly, such an applied conception still plays an important role in how children learn about numbers today, in kindergarten and elementary school.[3] What makes it particularly relevant to compare Frege's views with this conception is that he agrees with a central aspect of it: the priority it assigns to (certain) applications of arithmetic. As such, it is much closer to Frege's own views than crude formalist or psychologist ones. Indeed, one can see Frege's approach as a natural extension, or update, of it (as elaborated further below, in §1.2).

Of course, Frege also disagrees with the pluralities conception in important respects. There are two main areas of disagreement. First, he finds the way, or ways, in which talk about "pluralities", "groups" or "multitudes", as well as talk about their "grouping", "combining" or "adding", has been understood problematic. Second, he believes that such an understanding of arithmetic terms and statements, even if accepted in itself, does not provide us with a conception adequate to the science of arithmetic (as opposed to the ordinary use of arithmetic language in simple applications). Let me say a bit more about both, so as to set the stage for our subsequent discussion of Frege's alternative.

For Frege, the main problem with the notion of "plurality" ("multitude", "group", also "totality", "collection", even "set"), as used before and during his time, is that it is left unclear how concrete or abstract the relevant pluralities are supposed to be. From a contemporary point of view, it is natural to think of them as (finite) sets, where sets are understood to be abstract objects. However, such an understanding was not available during Frege's time, or at least Frege did not find it in the literature in any clear form. Instead, pluralities were sometimes taken to be concrete agglomerations or heaps, with a location in space and time, with physical properties such as

extension, weight, and color, and accessible empirically just like physical objects. Also, often they were thought to be composed of their constituents the way in which a whole is composed of its parts. In other words, there was a tendency to understand what a "plurality" is not in an abstract set-theoretic, but in a concrete mereological sense; or perhaps better, these two understandings were not separated carefully yet. Similar criticisms apply to the corresponding operations of "grouping", "combining", or "adding".

As to Frege's second area of disagreement, it is true that the pluralities conception seems adequate for simple arithmetic statements of the form "2 + 3 = 5", as used in ordinary applications, i.e., it seems possible to analyze them as involving "common names", etc. However, things change when we move on to more complex arithmetic statements, especially ones we would express today—following Frege—by using quantifiers (including higher-order quantifiers, or equivalently quantification over sets of numbers). How is the pluralities conception to be applied, or extended, to such cases? Moreover, already a simple arithmetic statement such as "2 is a prime number" is naturally understood to be one in which a property (to be prime) is attributed to an object (the number two). Finally, even basic applied statements such as "Jupiter has four moons" can be analyzed in such a way that the number four plays the role of an object in them ("The number of moons of Jupiter = 4"). If such Fregean observations are correct, then a comprehensive and scientific understanding of the concept of number requires treating numerical terms ("2", "2 + 2", "the number four", "the number of *F*s", etc.) as singular terms, thus numbers as objects.

1.2 In response: Basic moves and core ideas[4]

Responding to the problems he finds in connection with the notion of "plurality" in the literature of his time, Frege's first basic move is to replace it with the notion of a "concept" (itself in need of clarification, as he realizes himself later). This replacement brings with it two Fregean observations, closely related to each other: First, the relevant concepts can be recognized to account for the needed individuation in numbering, an aspect often left obscure in earlier views. For example, when we say "four companies" (in an army), the unit of what is being numbered is provided by the concept "company"; while when we say "five hundred men" (in the same army), an alternative unit is provided by the concept "man". Second, numerical statements can now be analyzed as statements about concepts. Thus, "Jupiter has four moons" turns out to be a claim about how many objects fall under the concept "moon of Jupiter", namely exactly four. Taken together, this accounts for both the relativity (relative to a concept) and for the objectivity (given a concept) of numerical statements.

A second basic move by Frege, parallel to the first, is to direct attention away from concrete, physical relations and operations on pluralities, and towards logical functions on concepts. In particular, the relation of equinumerosity (as underlying statements such as "The number of moons of Jupiter = 4" or "2 + 2 = 4"), earlier usually understood in physical or at least spatio-temporal terms, reveals itself now as analyzable in terms of the existence of a bijective (1 – 1 and onto) function between the objects falling under the corresponding concepts. Similarly, the "collecting together" or "adding" (relevant for "2 sheep and 3 sheep" or "2 + 3") is now understood in terms of logical functions, especially a logicized version of the successor function.

With these first two Fregean moves, the door for logicism is opened. A third, more systematic move then backs them up, by providing a precise, general framework for the approach. This is Frege's introduction of his new logic—a form of higher-order logic—powerful enough to incorporate the concepts and functions just mentioned. Note also that, within this framework, a series of "numerical concepts" become available: the second-order concepts "zero-ness", "one-ness", "two-ness", etc.; e.g., "two-ness" can be defined as follows (in contemporary notation): $\exists x_1 \exists x_2 (X(x_1) \wedge X(x_2) \wedge x_1 \neq x_2 \wedge \forall x_3 (X(x_3) \rightarrow (x_3 = x_1 \vee x_3 = x_2)))$ (where X is a unary second-order variable). This provides Frege with a precise, general way of analyzing statements such as "Jupiter has four moons" within his logic, namely (paraphrased in English): the concept of "moon of Jupiter" falls under the second-order concept of "four-ness".

With these initial moves, Frege is in a position to deal logically with all statements occurring in simple applications of arithmetic, thus making the appeal to pluralities superfluous. However, his bigger goal is to be able to handle not just such statements, but all arithmetic statements, including those occurring in the science of arithmetic of his day, i.e., higher-order number theory. As the most prominent example, Frege needs to find a way of analyzing logically the principle of mathematical induction. This task is, in fact, so central for Frege that it is the first he turns to after having put his new logical framework in place. His initial solution consists in a general analysis of the notion of "following in a sequence", or of the "ancestral relation", within higher-order logic.

There is still a further step, or jump, that is crucial for Frege. Namely, he wants to be able to conceive of numbers as objects. This is of special importance to him for at least four reasons (the first two of which we already encountered in his criticisms of the pluralities conception): First, Frege thinks that there are grammatical arguments to the effect that numerical terms, most basically "the number of *F*s", can and should be treated as singular terms. Second and centrally, he emphasizes that within mathematics, especially within the science of arithmetic, numbers play the logical role of objects. Third and more idiosyncratically, the elaboration of Frege's views about the fundamental difference between concepts and objects reveals that there are peculiar obstacles in referring to concepts (the "concept horse problem"). And fourth, within Frege's systematic

reconstruction of arithmetic, as spelled out in *Grundgesetze der Arithmetik* (Frege 1893/1903), the fact that numbers are treated as objects plays an important role in the proof that there are infinitely many natural numbers.

To take stock briefly, so far we have seen how Frege is led to the central role of concepts in the application of arithmetic, thus to a new logical analysis of numerical statements as statements about concepts. This also leads him to his new logical system, including a general theory of concepts and functions. And that system brings with it the second-order numerical concepts of "zero-ness", "one-ness", etc. At the same time, for Frege the natural numbers should not be thought of as concepts, but as objects, so that an identification of these numbers with the numerical concepts, attractive as it may seem in other respects, is ruled out. The crucial question now is: What kind of objects are numbers, if any; and how, more specifically, should we characterize them?

With respect to answering the first part of this question, Frege is guided by two general considerations. First, neither physical nor mental objects will do, both because we would need an infinite amount of them, which may not be available, and because this would make arithmetic depend on empirical or intuitive considerations in other problematic ways as well. Second, arithmetic reasoning has the interesting, but often unaccounted for, feature that it is completely general, i.e., applicable not just to the material world (like physics) or to mental phenomena (like psychology), nor merely to spatio-temporal and intuitable facts (like geometry), but beyond. In this respect it is rather like logic, since logical reasoning is also applicable not just in those restricted domains, but completely generally. Together, these considerations point towards conceiving of numbers as logical objects, if that is possible. Indeed, within the literature of Frege's time, and within the tradition he identifies with, there is a kind of logical objects that suggests itself: "extensions of concepts [*Begriffsumfänge*]" or, as Frege will also say in his later writings, "classes". Thus, why not identify numbers with classes?

Yet, the question remains: which classes exactly? Here the following two guiding ideas come in (the first at least implicitly, the second explicitly): First of all, the class that is to serve as a particular natural number (say the number two) should be related to all corresponding concepts (those under which exactly two objects fall) in some intimate and uniform way. Second, if we compare all the concepts corresponding to a particular number, it becomes apparent that they are all related to each other by being equinumerous, in the logical sense specified above. Now, within the mathematics of Frege's time—geometry and algebra, in particular—a technique is available that not only allows us to take into account both of these considerations, but also to identify numbers with classes, namely, the use of equivalence classes for introducing mathematical objects.[5] In addition, it seems natural to assume that the use of classes, including equivalence classes, can be built into Frege's new logic directly: by thinking about classes as extensions of concepts. A correspondingly enlarged logical system will, then, allow for a systematic logical foundation of that technique, including for the case that is central for Frege.

What we have been led to is the main technical move in Frege's logicist reconstruction of arithmetic: the construction of the natural numbers as equivalence classes under the relation of equinumerosity (obviously an equivalence relation). More particularly, we have been led to the use of equivalence classes of concepts (as opposed to equivalence classes of classes). Thus, the number two is identified with the class to which those concepts belong under which exactly two objects fall. Note that, along these lines, all the two-element concepts are both intimately and uniformly related to the number two: by being contained, as elements, in the relevant equivalence class. In addition, the number two turns out to be precisely the extension of the concept "two-ness", since it is all and only the two-element concepts that fall under that concept. In other words, natural numbers are now not only closely related to all the relevant first-order concepts, but also to the corresponding second-order numerical concepts.[6]

This is not yet the classic Frege-Russell conception of the natural numbers, but something close to it. We get the Frege-Russell conception itself if we replace the use of equivalence classes of concepts with that of equivalences classes of extensions or classes. Such a further move finds its motivation in two related observations: first, that all that matters in this context is to employ concepts as identified extensionally; second, that we might then as well use classes instead of concepts, since they are identified extensionally anyway and since this simplifies the construction slightly (more on that later).

Actually, to obtain a conception of the natural numbers here, as opposed to a conception of cardinal numbers more generally, one further ingredient is needed. We need to restrict ourselves to "finite" concepts or classes, respectively (in a non-circular way, i.e., not presupposing the natural numbers already). This brings us to the final basic move in Frege: the application of his initial logical analysis of "following in a sequence" in this particular context, i.e., to define the class of the natural numbers. It is defined as the smallest class that contains zero (understood as the equivalence class of "empty" concepts or classes) and is closed under the successor function (conceived of logically).

The classic statement of the Frege-Russell conception of numbers, as just described, is probably in Bertrand Russell's writings, beginning with "The Logic of Relations" (Russell 1901) and *Principles of Mathematics* (Russell 1903), later also in *Introduction to Mathematical Philosophy* (Russell 1919). Actually, even putting aside Frege's work for the moment, this conception seems to have been in the air already before Russell's writings. For example, the mathematician Heinrich Weber proposes essentially the same conception, independently of both Frege and Russell, in an 1888 letter to Richard Dedekind (published posthumously).[7] In any case, it should be clear by now

that this conception is well motivated, both historically and systematically—I can still feel its considerable attraction today.

2.1 The development of Frege's ideas: Before Russell's antinomy

So far we have discussed the relationship of Frege's conception of the natural numbers to earlier such conceptions, in particular to the pluralities conception, thus bringing to the fore its core ideas and their motivations. Now I want to turn to the development of Frege's position in his writings, i.e., the order in which his basic views were introduced, also the ways in which he responded to questions and problems along the way.

My point of departure will be *Begriffsschrift* (Frege 1879), the work in which Frege begins to address the connection between logic and arithmetic explicitly. Towards the end of its preface, he writes:

> Arithmetic . . . was the starting point of the train of thought that led me to my Begriffsschrift. I therefore intend to apply it to this science first, seeking to provide further analysis of its concepts and a deeper foundation of its theorems. I announce in the third Part some preliminary results that move in this direction. Progressing along the indicated path, the elucidation of concepts of number, magnitude, etc., will form the object of further investigations, to which I shall turn immediately after this work.
>
> (Frege 1997, pp. 51–52)

In this passage, three points come up that are important for present purposes. First, there is the introduction, in the book *Begriffsschrift*, of Frege's new logical system, his Begriffsschrift, intended from the beginning to be applied in providing a new foundation for arithmetic, as he says explicitly. Second, Frege points towards some specific "preliminary results that move in this direction". These concern his logical analysis of "following in a sequence", to be used later in his logical analysis of mathematical induction. Third, Frege announces that in subsequent work he will elucidate "the concept of number", thus indicating that he has not done so in this first book.

Indeed, in *Begriffsschrift* no definition or construction of the natural numbers, or of the basic arithmetic functions and relations, is attempted. Frege still leaves their nature completely open—except for one aspect: his analysis of "following in a sequence" is set up in such a way that it is objects that are to be arranged sequentially. That suggests that, in the intended application of this analysis to the natural numbers, the numbers will play the role of objects as well.[8] Then again, no emphasis is placed on a sharp distinction between concepts and objects at this point, at least not explicitly. Moreover, no theory of extensions or classes is provided yet. The logical system introduced in *Begriffsschrift* contains only what one may call the "purely logical" or "inferential" part of higher-order logic, without any means for constructing extensions or classes.

The work in which Frege's promised "elucidation of the concept of number" is presented for the first time is, of course, *Die Grundlagen der Arithmetik* (Frege 1884), published five years after *Begriffsschrift*. It is also in the introduction to that book that Frege first formulates the basic principle "never to lose sight of the distinction between concepts and objects" (p. x) (together with two other guiding principles). This principle's immediate and main application in the book is, then, in characterizing the natural numbers as "self-subsistent objects" (p. 72 etc.), not as concepts. As mentioned earlier, this rules out the identification of the natural numbers with the second-order numerical concepts of "zero-ness", "one-ness", "two-ness", etc., which occur naturally in the logical system of *Begriffsschrift*.

It is also in *Grundlagen* that the construction of the natural numbers as equivalence classes of concepts is proposed for the first time. In fact, this proposal constitutes the core of the book's non-polemical part. Frege starts by defining cardinal numbers in general as follows: "The number which belongs to the concept *F* is the extension of the concept 'equinumerous to the concept *F*'" (Frege 1884, p. 85).[9] He goes on: "0 is the number which belongs to the concept 'not identical with itself'" (p. 87); "1 is the number which belongs to the concept 'identical with 0'" (p. 90); etc. To be sure, Frege himself does not present this construction as the natural outgrowth of the pluralities conception; but a focus on the (cardinal) application of numbers, as shared by the pluralities conception, is clearly guiding him. Beyond that, Frege's analysis of numerical statements as statements about concepts is argued for explicitly in *Grundlagen*, in terms of the reasons mentioned above.[10]

As we just saw, Frege appeals to classes, or rather to "extensions of concepts", in the central construction of *Grundlagen*. However, this appeal seems still somewhat tentative, and is not backed up by a systematic theory of such extensions. Indeed, in a tantalizingly pregnant footnote, occurring just after the equivalence class construction, he writes:

> I believe that for "extension of the concept" we could write simply "concept". But this would be open to the two objections:
>
> 1. that this contradicts my earlier statement that the individual numbers are objects, as is indicated by the use of the definite article in expressions like "the number two" and by the impossibility of speaking of ones, twos, etc. in the plural, as also by the fact that the number constitutes only an element in the predicate of a statement of number;
> 2. that concepts can have identical extensions without themselves coinciding. I am, as it happens, convinced that both these objections can be met; but to do this would take us too far afield for present purposes.
>
> I assume that it is known what the extension of a concept is.
>
> (Frege 1884, p. 80)

Cryptic as it is, I take this remark to establish at least three points relevant for us:[11] First, Frege's views on how to think about, or at least how to present, the distinction between concepts and objects have not yet reached their mature form in *Grundlagen*. Second, the notion of concept is, in his own view, in need of further clarification at this point, especially with respect to the question of whether to think of the identity of concepts intensionally or extensionally. But also third, he takes the notion of the extension of a concept to be given and sufficiently well understood, in some traditional sense. All three points, or their further clarification, become the subject of subsequent writings.

The writings in question are: "Funktion und Begriff" (Frege 1891), "Über Sinn und Bedeutung" (Frege 1892a), "Über Begriff und Gegenstand" (Frege 1892b) and the two volumes of *Grundgesetze der Arithmetik* (Frege 1893/1903). In the first two articles, Frege starts to use his metaphors of "saturated vs. unsaturated" and "complete vs. incomplete" to clarify the distinction between objects and concepts. He also introduces his conception of concepts as truth-valued functions (thus essentially as the characteristic functions of their extensions). Built into that conception is his introduction of the two truth values as logical objects, as well as his general decision to use extensional criteria of identity for functions, including concepts. Connected with the latter is, moreover, the introduction of his famous "sense-reference [Sinn-Bedeutung]" distinction. One benefit of making that distinction is that it opens up the possibility of seeing functions, conceived of extensionally, as the referents of function names, thus concepts as the referents of concept names, while the "intensional aspect" often associated with functions and concepts is separated out and incorporated into the sense of the relevant names.[12] In "Über Begriff und Gegenstand", the third article from this period, Frege then defends his fundamental distinction between concepts and objects further against certain objections. This leads him to the declaration that that distinction can only be elucidated informally, since uses of the phrase "the concept *F*", as in "the concept horse", will strictly speaking not allow us to refer to concepts.

With those distinctions and decisions in place, Frege has substantially clarified several of his crucial notions. However, he still has not provided us with a systematic account of "extensions of a concept" or "classes". Such an account is a main goal of *Grundgesetze der Arithmetik*, via an extension of the logical system from *Begriffsschrift.* In particular, Frege now adds a theory of "value-ranges" to his logic, in such a way that extensions or classes are covered as a special case. The central step in this connection is to add a logical axiom that governs the use of value-range terms: Frege's Basic Law V. As restricted to extensions, it says: $\grave{\varepsilon}F(\varepsilon) = \grave{\varepsilon}G(\varepsilon) \leftrightarrow \forall x(F(x) \leftrightarrow G(x))$ (where "$\grave{\varepsilon}F(\varepsilon)$" is the Fregean term used for the extension of the concept F etc.). Crucially and infamously, this law (in conjunction with Frege's other basic laws and rules of inference) implies the existence of an extension for any concept, thus leading to Russell's antinomy.

At this stage—after Frege (thinks he) has accounted for extensions or classes systematically, within a logical theory—he comes back to the equivalence class construction for the natural numbers. Relative to *Grundlagen*, this construction is now modified in one respect: Frege no longer uses equivalence classes of concepts, but equivalence classes of classes (*Grundgesetze*, §§40–43). Thus now we are presented with the classic Frege-Russell conception, within the framework of Frege's mature logic. It is followed by detailed treatments of various arithmetic notions and propositions, including mathematical induction. Here Frege incorporates, in an explicit and formal way, all of his earlier insights as mentioned above.

Why, once more, does Frege make the shift from equivalence classes of concepts to equivalence classes of classes, especially at this point? He is not very explicit about it; indeed, the shift can easily be overlooked, since it is buried under technical details. The following two reasons suggest themselves: First, his decision to understand concepts extensionally, as carried over from "Funktion und Begriff", makes it possible to move easily back and forth between concepts and their extensions; they now correspond to each other one-to-one. Second, using extensions or classes instead of concepts in the construction makes the technical development of Frege's view slightly simpler; basically, it now suffices to work with extensions just for first-level concepts.[13]

Whatever the precise reasons for the shift, in *Grundgesetze*, unlike in *Grundlagen*, Frege feels fully justified in using extensions; he also uses the term "class" more and more, sometimes even "set".[14] This feeling turns out to be illusory, of course—the problematic nature of Basic Law V will soon become evident. At the same time, the following two points can be made in (partial) defense of Frege: First, while he probably derived his use of the equivalence class construction from earlier such constructions, as indicated above, he was one of the first to emphasize the need for providing this technique with a rigorous foundation; and he was the first, as far as I am aware, to attempt providing such a foundation in the form of an axiomatic theory of classes, more specifically a version of type theory. Second, in the course of analyzing the foundations of new mathematical techniques it happens not infrequently that limits to their range of applicability become apparent that were very hard, perhaps even impossible, to detect beforehand.

Unfortunately for Frege, his own application of the equivalence class construction in logicizing arithmetic turns out to lie outside the range for that technique.

2.2 The development of Frege's ideas: After Russell's antinomy

So far I have traced the rise of the Frege-Russell conception of the natural numbers in Frege's works. What remains to be examined is its fall, or the ways in which Frege reacted to the announcement that his logical system is subject to Russell's antinomy.

Russell informed him of that fact in a letter dated June 16, 1902 (Russell 1902a). Frege's response, in a letter from June 22 (Frege 1902a), is the following:

> Your discovery of the contradiction has surprised me beyond words and, I should almost like to say, left me thunderstruck, because it has rocked the ground on which I meant to build arithmetic. It seems accordingly that the transformation of the generality of an equality into an equality of value-ranges (§9 of my *Grundgesetze*) is not always permissible, that my law V (§20, p. 36) is false, and that my explanation in §31 do not suffice to secure a reference for my combination of signs in all cases. I must give some further thought to the matter. It is all the more serious as the collapse of my law V seems to undermine not only the foundations of my arithmetic but the only possible foundation of arithmetic as such. And yet, I should think, it must be possible to set up conditions for the transformation of the generality of an equality into an equality of value-ranges so as to retain the essentials of my proof. Your discovery is at any rate a very remarkable one, and it may perhaps lead to a great advance in logic, undesirable as it may seem at first sight.
>
> (Frege 1997, p. 254)

He also adds an appendix to volume II of *Grundgesetze*, which reads similarly:

> Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished.
>
> This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion. It is a matter of my Axiom (V). I have never disguised from myself its lack of the self-evidence that belongs to the other axioms and that must properly be demanded of a logical law. And so in fact I indicated this weak point in the Preface to Vol. I (p. VII). I

> should gladly have dispensed with this foundation if I had known of any substitute for it. And even now I do not see how arithmetic can be scientifically established; how numbers can be apprehended as logical objects, and brought under review; unless we are permitted—at least conditionally—to pass from a concept to its extension. May I always speak of the extension of a concept—speak of a class? And if not, how are the exceptional cases recognized? Can we always infer from one concept's coinciding in extension with another concept that any object that falls under the one falls under the other likewise? These are the questions raised by Mr. Russell's communication.
>
> *Solatium* [sic] *miseris socios habuisse malorum*. I too have this comfort, if comfort it is; for everybody who in his proofs has made use of extensions of concepts, classes, sets, is in the same position as I. What is in question is not just my peculiar way of establishing arithmetic, but whether arithmetic can possibly be given a logical foundation at all.
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> (*Ibid*., pp. 279–280)

In addition, Frege proposes a weakening of Basic Law V in the same appendix. This weakening is supposed to save his system from contradiction, but still allow for most parts of his logicist project, in particular the Frege-Russell construction.

Several points are noteworthy, for present purposes, in this initial response of Frege's to Russell's antinomy. First of all, Frege immediately recognizes the significance of Russell's result, including the fact that it forces him to make some kind of change to the way in which value-ranges, thus extensions or classes, are introduced. His first stab at making such a change is to keep working with extensions for all concepts, but to fiddle with Basic Law V, i.e., to modify the logic of extensions slightly so as to avoid the antinomy. And why does he attempt to do that? He still thinks that it "must be possible to set up conditions for the transformation of the generality of an equality into an equality of value-ranges so as to retain the essentials of my proof". Also, doing so seems to him the only way in which "arithmetic can be scientifically established", in particular the only way to provide it with "a logical foundation". Even more specifically, how else could numbers "be apprehended as logical objects", if not by identifying them with classes?[15]

Not long thereafter Frege realizes, however, that this initial proposal won't work.[16] During the following years, indeed until his retirement in 1918, what follows is a silent period, at least on the topic at issue. Frege's only publications from that period are some articles on the foundations of geometry, in response to Hilbert's work, and a few short polemics against formalist theories of arithmetic. This makes it hard to see what his subsequent, more considered reaction to the antinomy is, also when exactly he

gives up on the modification of Basic Law V proposed initially. If we want to get insight into his further thoughts on the issue, we thus have to go beyond his published works; we have to turn to a few pieces in his *Posthumous Writings*, as well as two more unusual sources: notes taken by Rudolf Carnap, in 1910–1914, as a student in Frege's classes on logic and the foundations of mathematics (Frege 1996 and 2004) and the report of a conversation, in 1913, between Frege and Ludwig Wittgenstein (Geach 1961).

What the lecture notes from Frege's classes reveal, in our context, is the following: Frege's way of avoiding Russell's antinomy in 1910–1914, while presenting his logic to students like Carnap, is simply to leave out the part of his logical system that has to do with classes, or more generally with value-ranges. In particular, Basic Law V does not make any appearance in these notes, nor does any modification of it. Instead, Frege restricts himself to the inferential part of higher-order logic, as he did initially in *Begriffsschrift*. All his other mature clarifications and distinctions, e.g. those between concepts and objects, sense and reference, and his extensional understanding of concepts, remain in place, though. One further aspect of Frege's lectures, as recorded by Carnap, is also noteworthy. Namely, the second-order numerical concepts of "zero-ness", "one-ness", etc. come up very explicitly.[17] This may make one wonder whether Frege now wants to identify numbers with those concepts. But that is not the case; in a few asides he still treats numbers as objects, not as concepts. Then again, no elaboration of what the nature of numbers is supposed to be now is given.

It seems, therefore, that in 1910–1914 Frege is still holding on to the view that numbers are objects, but has given up on the theory of classes as a means for constructing them. This impression is confirmed by a conversation Frege had with Wittgenstein in 1913. Peter Geach reports Wittgenstein relating this conversation to him later as follows:

> The last time I saw Frege, as we were waiting at the station for my train, I said to him: 'Don't you ever find *any* difficulty in your theory that numbers are objects?' He replied 'Sometimes I *seem* to see a difficulty—but then again I don't see it.'
>
> (Geach 1961, p. 130)

Note here that Wittgenstein is not asking about Frege's theory of numbers as logical objects, but as objects more generally. Whether Frege is still holding on to viewing numbers as logical objects at this point is not made clear, although the form of Wittgenstein's question could indicate that he has given that up now, perhaps together with rejecting classes. For Wittgenstein himself, as spelled out in his *Tractatus Logico-Philosophicus*, numbers are, of course, not even objects, but "exponents of operations".[18]

Actually, even the claim that numbers should be treated as objects may have come into doubt for Frege during this general period, as his tentative "sometimes I *seem* to see a difficulty—but then again I don't see it" in response to Wittgenstein already suggests. Further evidence for such doubt, or ambivalence, can be found in Frege's Nachlass, e.g., in notes written, in 1919, for the historian of science Ludwig Darmstaedter. In these notes Frege asserts once more: "In arithmetic a number-word makes its appearance in the singular as a proper name of an object of this science" (Frege 1919, p. 256) and concerning the second-order numerical concepts he adds: "We do not have in them the numbers of arithmetic; we do not have objects, but concepts" (pp. 256–257). But he also asks: "Could the numerals help to form signs for those second-order concepts, and yet not be signs in their own right?" (p. 257). Then again, Frege does not follow up on the suggestion contained in that question, which would have meant giving up seeing numbers as objects and treating them in some more contextual way.

With the notes for Damstaedter written in 1919, one year after his retirement from the University of Jena, we have entered the last phase of Frege's intellectual development. During this phase, he again publishes several articles, especially a series of essays on the philosophy of logic, starting with "Der Gedanke" in 1918. More importantly for us, towards the very end of his life Frege writes various notes to himself, published posthumously, in which he returns to questions about the foundations of arithmetic. These include: "Zahl" (Frege 1924), "Erkenntnisquellen der Mathematik und mathematischen Naturwissenschaften" (Frege 1924/1925a), "Zahlen und Arithmetik" (Frege 1924/1925b), and "Neuer Versuch der Grundlegung der Arithmetik" (Frege 1924/1925c).

What these very late notes establish is that Frege sees little promise any more, at this point, for his logicist reconstruction of arithmetic. As he writes:

> My efforts to throw light on the questions surrounding the word 'number' and the words and signs for individual numbers seem to have ended in complete failure. Still, these efforts have not been wholly in vain. Precisely because they have failed, we can learn something from them. The difficulties of these investigations are often greatly underrated.... These investigations are especially difficult because in the very act of conducting them we are easily misled by language.
>
> (Frege 1979, pp. 265–266)

Frege locates the source for the "complete failure" in his earlier theory of extensions:

> One feature of language that threatens to undermine the reliability of thinking is its tendency to form proper names to which no object corresponds. . . . A particularly noteworthy example of this is the formation of a proper name after the pattern of 'the extension of

> the concept *a*' . . . . I myself was under this illusion when, in attempting to provide a logical foundation for numbers, I tried to construe numbers as sets.
>
> (*Ibid.*, p. 269)

It is striking that extensions are now seen as an "illusion", created by a misleading feature of language. On the other hand, Frege goes on to reaffirm his belief that, for scientific purposes, numbers need to be seen as objects and as distinct from the corresponding numerals; as he puts it: "For a number, by which I don't want to understand a numerical sign, appears in mathematics as an object, e.g., *the* number 3" (p. 271, *cf.* pp. 265–266).

What, at this late stage, is the alternative to viewing numbers as classes, given that they are still to be treated as objects? Frege's answer is hinted at in the following remark:

> The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis—a geometrical one in fact—so that mathematics in its entirety is really geometry.
>
> (*Ibid.*, p. 277)

And what does Frege have in mind when he wants to conceive of arithmetic as part of geometry; in particular, how are we to conceive of numbers then? He states briefly: "Right at the outset I go straight to the final goal, the general complex numbers" (p. 279). That is to say, the complex numbers are to be constructed geometrically, and the natural numbers are then, presumably, to be identified as specific complex numbers.

Let me add one final observation in this connection. Clearly, with the suggestion to base arithmetic on geometry Frege has given up logicism. This is a major reversal in his views. Why exactly does he feel forced to make that reversal? The quick answer is that his logical system has turned out to be inconsistent, and that no satisfactory repair has occurred to him. A related, but deeper answer is hinted at in the following remark: "It seems that [the logical source of knowledge] on its own cannot yield us any objects" (*ibid.*, p. 279). Now, this remark can be understood in two slightly different ways: Either for Frege classes, or value-ranges more generally, are the only entities that could possibly have counted as logical objects, so that with their demise no such objects remain (except perhaps for truth values, assuming that they are not necessarily to be identified with value-ranges). Or for him any kind of logical objects one might want to appeal to is in need of a rigorous, systematic justification, in particular in terms of how we apprehend such objects, but after the failure of *Grundgesetze* Frege no longer sees any way of giving such a justification, at least not in purely logical terms. The former interpretation might be supported thus:[19] From *Begriffsschrift* to his last writings, Frege thinks of logic as concerned with concepts (or functions more generally), and extensions (value-ranges more generally) are tempting for him because of their intimate connection with concepts (similarly for truth values). Support for the latter interpretation may be found in remarks such as the following, from another letter to Russell, dated July 28, 1902 (Frege 1902b), which also echoes the appendix to *Grundgesetze*, volume II (quoted above):[20]

> I myself was long reluctant to recognize value-ranges and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation. The question is: How do we apprehend logical objects? And I have found no other answer to it than this: We apprehend them as extensions of concepts, or more generally, as value-ranges of functions. I have always been aware that there are difficulties connected with this answer, and your discovery of the contradiction has added to them. But what other way is there?
>
> (Frege 1980, pp. 140–141, translation slightly altered)

It is hard to be certain which of these two interpretations is correct, since Frege says so little about the issue. Maybe they cannot even be kept separate in the end?

3.1 Neo-Fregean rejoinders: Working without classes

Towards the end of his life, Frege has clearly given up on the project of reducing arithmetic to logic, especially via a logical theory of classes. Instead, he proposes to reduce arithmetic to geometry. This last proposal has not been explored much in the Frege literature, nor in the philosophy of mathematics more generally.[21] It is not hard to explain why. As a position in itself, the reduction of any part of mathematics to geometry has foundational significance only if it can be argued that geometry has some special status. In Frege's view it did, in fact, have such a status, since he thought of geometry as based on Kantian intuition. However, this kind of view has lost its appeal for most philosophers of mathematics today, for various reasons.

Moreover, even from Frege's point of view, as discussed so far, two questions arise immediately: First, what about the universality of arithmetic; is this feature compatible with a reduction of arithmetic to geometry or is it to be given up (since geometry is restricted to what is intuitive)? Second, what about the close connection between the nature of numbers and their applications, as highlighted in the Frege-Russell conception; do we have to give that up as well (since the usual geometric constructions of the complex numbers do not incorporate it)?[22] In his late writings, Frege doesn't say anything to answer the first of these questions. In connection with the

second, a few brief remarks suggest that he seems willing to bite the bullet; thus he writes:

> Now of course the kindergarten-numbers appear to have nothing whatever to do with geometry. But that is just a defect in the kindergarten-numbers. . . . Counting, which arose psychologically out of the demands of business life, has led the learned astray.
>
> (Frege 1979, p. 277)

In other words, Frege is now willing to separate the basic applications of the natural numbers, as learned from kindergarten on, from the account of their nature. This is another radical step for Frege, a change of mind that cuts quite deep.

If the steps considered by Frege in his last writings seem too radical, and not attractive from a contemporary point of view, this is not the end of the story. Recently hope has been rekindled that we can go back to Frege's original project, his logicist reconstruction of arithmetic, and revive it by new means, perhaps even "neo-logicist" means. The goal here is not just to propose some *ad hoc* modification that saves Frege's approach from contradiction, but to arrive at a "neo-Fregean" position that is attractive in itself. In the remainder of this paper, I want to compare five such proposals. The first two (to be discussed in the rest of the present section) involve ideas that Frege was well aware of, or that were at least within his reach, but that, for some reason or other, he didn't pursue. Both of them also involve giving up any reduction of arithmetic to a theory of classes, just like Frege's late proposal to reduce arithmetic to geometry.

Suppose then, for the moment, that we discard Frege's theory of classes. Suppose, at the same time, that we still want to work within (the remaining parts of) his logical system. What we need is something else that can play the role of the natural numbers. Now, we saw above that within Frege's logic the numerical concepts "zero-ness", "one-ness", etc. occur naturally. In particular, they occur already in his *Begriffsschrift* system, before the introduction of extensions or classes, and again in the logic of his 1910–1914 lectures, after he has discarded classes. In addition, these concepts are logical entities (certain higher-order functions); and they are closely related to the ordinary applications of arithmetic (see above). Indeed, within Frege's logical system they are the entities closest to the problematic equivalence classes that we can still get (since the equivalence classes, if they existed, would be their extensions). But then, why not identify the natural numbers with these concepts? In the literature, this idea has been explored by, amongst others, David Bostock and Harold Hodes.[23]

Frege's own reasons for resisting such a move have come up already: first, his insistence on a strict distinction between objects and concepts, coupled with his conviction that numbers fall on the side of objects; and second, the obstacles he finds in connection with referring to concepts. However, perhaps one can argue, against Frege, that the strict separation of objects and concepts has to be given up; or perhaps one can reanalyze all numerical statements in such a way that numbers play the role of concepts after all, in general. Maybe one can also show that it is possible to refer to concepts with phrases such as "the concept *F*", i.e., that the concept-horse problem is a non-problem. Even assuming all of that, there is still a remaining problem. Namely, we need to be able to establish that there are infinitely many natural numbers. Frege's own (attempted) proof of that result in *Grundgesetze* relies on treating numbers as objects, more particularly as classes. Now that classes have been given up, how else could we possibly establish it, especially based on logic alone? That question leads to many thorny issues concerning the existence and identity of concepts.

A second recent proposal for how to revive Frege's logicism doesn't appeal to second-order numerical concepts, but to a corresponding numerical function and its values. This proposal makes central use of "Hume's Principle", which says (in contemporary notation): $\#F = \#G \leftrightarrow F \sim G$ (where "#" stands for the second-order numerical function in question and "$\sim$" for the second-order relation of equinumerosity, defined within higher-order logic).[24] It has been defended most vigorously, as a neo-logicist position, by Crispin Wright and Bob Hale, but discussed also by George Boolos, John Burgess, William Demopoulos, Kit Fine, Richard Heck, and others. Indeed, it is this kind of approach that is largely responsible for the recent revival of interest in Frege's philosophy of mathematics. As has also become clear (to some degree at least), it can be extended beyond arithmetic, by adopting more general "abstraction principles".[25]

Such an approach gets part of its motivation, especially as a "neo-Fregean" position, from the following two observations: First, in Frege's treatment of the natural numbers in *Grundgesetze* the problematic Basic Law V is used essentially only to establish Hume's Principle; the subsequent results are all derived from that principle, within second-order logic. Second and more specifically, the Dedekind-Peano Axioms can be derived from Hume's Principle within second-order logic; indeed, Frege himself essentially does so ("Frege's Theorem").[26] A third, post-Fregean insight is then added, namely: The system consisting of second-order logic and Hume's Principle can be shown to be consistent relative to set theory, thus not subject to Russell's antinomy; indeed, it is equiconsistent with second-order Peano arithmetic. In other words, large parts of Frege's technical work in *Grundgesetze* are actually safe and valid. In addition, we can note that the numerical functions # is closely related to the ordinary applications of arithmetic—as closely, one may want to say, as the equivalence classes in the Frege-Russell conception. What that means is that, in addition to several technical developments, a central part of Frege's underlying motivation can also be preserved.[27]

In fact, the resulting position can again be seen as a natural outgrowth of (what is right in) the pluralities conception.

Exploring the consequences of Hume's Principle and its generalizations within higher-order logic has certainly proved fruitful, both in terms of new technical results and lively philosophical discussions of its neo-logicist aspirations (related to the claim that Hume's principle should be accepted as "quasi-definitional"). In our context, however, the following points need to be added: First, it is fairly clear that Frege was aware of this kind of proposal. His closely related discussion of the notion of the direction of a line in *Grundlagen*, §64, indicates that, as do some remarks in his correspondence with Russell (especially in Frege 1902b). Second, not only did Frege refrain from adopting the proposal, he even actively rejected it. One reason for that rejection was the "Julius Caesar problem", related to the fact that the principles in question do not, in themselves, determine all identities involving numbers. This is by now a familiar problem, and various post-Fregean solutions for it have been proposed (although no general agreement on its solution, or even on the precise nature of the problem, has been reached). A deeper, though not unrelated, reason may have been Frege's conviction that principles such as Hume's do not, in themselves, give us enough to "apprehend" logical objects.[28] Beyond that, Hume's principle may not have qualified as a basic logical principle for Frege because of its perceived lack of complete generality and *ad hoc* nature.[29]

Without being able to explore the issue in all detail here, I would like to make three further observations in this connection. Note, to begin with, that relying centrally on the numerical function # ties the number two, say, directly to all the two-element concepts *F* (all those falling under the concept "two-ness"). In this respect the resulting position is, again, quite "Fregean". However, in doing so terms of the form "#*F*" are treated as primitive, undefined terms (whereas in Frege's original proposal they are defined). From a historical perspective, this procedure is reminiscent of the "definitions by abstraction" discussed by Bertrand Russell in *Principles of Mathematics*, in connection with the work of Guiseppe Peano and his school (Russell 1903, pp. 114–115).[30] Russell rejects such "abstraction", of course, and, like Frege, replaces them precisely with the equivalence class construction, for reasons that may be worth reconsidering. Apart from that, it appears now that the proposal by Wright, Hale, and others is at its core more "neo-Peanesque" (if there is such a word) than "neo-Fregean".

Second, note that the objects to which terms of the form "#*F*" are taken to refer, along these lines, are different from classes. They are supposed to be distinct abstract, or even logical, objects. However, we saw above that classes, or value-ranges more generally, are the only logical objects Frege ever relied on in trying to reconstruct the natural numbers. Perhaps he even took them to be the only objects that could possibly count as logical objects, or at least the only such objects for which a rigorous and systematic justification seemed available. If so, then here we have another respect in which the proposal under consideration is quite un-Fregean.[31] Third, note that the objects introduced as the referents of "#F" etc., precisely insofar as they are primitive logical objects only characterized by Hume's Principle, do not seem to have any intrinsic properties (they have no elements etc.). What that points towards is a possible, but so far unexplored, connection between this kind of view and certain structuralist views about the natural numbers, especially Richard Dedekind's logical structuralism.[32] This is a third aspect in which the position seems quite un-Fregean.

3.2 Neo-Fregean rejoinders: Rehabilitating classes

While both Frege's late proposal (natural numbers as geometrically conceived complex numbers) and the two neo-Fregean proposals just considered (numbers as higher-order concepts and numbers as primitive abstract objects) avoid any appeal to classes in their reconstructions of arithmetic, the next three will bring back classes again, both in themselves and in connection with arithmetic. The common goal for them is to modify Frege's Basic Law V in such a way as to both make it consistent with the rest of Frege's logic and allow for the reconstruction of all, or large parts, of mathematics in terms of classes. In that respect, all three are aligned with what Frege attempted in his initial, but failed, response to Russell's antinomy. The differences between them, as well as relative to Frege's original rescue attempt, lie in how exactly that modification is to be effected.

One basic way of modifying Frege's theory of classes is by introducing predicative restrictions on which classes exist, or even on which underlying concepts exist. The classic version of such a proposal is Russell's ramified theory of types.[33] But recently other versions—based more closely on Frege's work—have also been studied by, among others, John Burgess, Fernando Ferreira, Alan Hazen, Richard Heck, Øystein Linnebo and Kai Wehmeier.[34] One part of the motivation for such proposals is the diagnosis, voiced most prominently by Michael Dummett, that the real source of the problem with Frege's system is his use of impredicatively constructed extensions or concepts.[35] Another part of the motivation, especially for some of the more recent proposals, is that relative consistency proofs for predicative subsystems of Frege's original theory can be given, subsystems within which at least some parts of mathematics can be reconstructed. And of course, predicative approaches in the foundations of mathematics have attracted attention more generally, from Hermann Weyl to Solomon Feferman and beyond.

Clearly the exploration of such avenues has, once more, led to many interesting results, especially of a technical nature. On the other hand, this kind of approach has some immediate limits that, especially from a Fregean point of view, must appear as drawbacks. Starting with Russell's work it has, in particular, become apparent that not all of classical mathematics can

be reconstructed within a strictly predicative system (without additional axioms such as the Axiom of Reducibility). And even if we restrict ourselves just to arithmetic, there are problematic aspects: in Russell's system we have to rely on a controversial Axiom of Infinity (controversial especially as a logical axiom) to be able to construct all the natural numbers; a duplication of these numbers occurs on each type level; and they turn out not to be full-fledged objects in the end, but only quasi-objects (in a "no-classes theory of classes").[36] Such features seem in clear conflict with Frege's original goals. Then again, suitably restricted versions of Frege's original proofs, and even of his equivalence class construction, can be shown to work along such lines.

Other basic ways in which one can try to modify Frege's logic, in particular his theory of classes, consists of restricting which concepts determine classes not by predicative, but by other means. Taking a cue from Zermelo-Fraenkel set theory one can, in particular, introduce a "limitation of size" principle, with the effect that only concepts that are "small" determine classes, but not those that are "large". Actually, a number of different variants of such a principle have been proposed, starting with George Boolos' "New V", a modification of Frege's original Basic Law V.[37] Alternatively, one can introduce a "Reflection Principle" to get similar restrictive effects, as established recently by Harvey Friedman (who has explored such ideas in a more general context).[38] A main attraction of such proposals is that they allow for the resurrection of Frege's theory of classes, within higher-order logic, by means of just one relatively simple modification of Basic Law V. Also, such modifications can leave the theory essentially as powerful as ZF set theory; thus they confine it far less than predicative modifications.

Once again, the investigation of such updates for Frege's system has been, and continues to be, fruitful in leading to various new technical results. From a philosophical perspective, they are especially attractive if one starts from the following two basic assumptions: one's goal is to develop a theory of classes in the sense of extensions of concepts and one finds the "limitation of size" idea well-motivated. However, from the point of view developed in the present paper there is again an immediate problem, or at least a consequence that should be noted. Namely, if we attempt to repeat Frege's original construction of the natural numbers within such a modified system, it becomes clear right away that that isn't possible; since Frege's equivalence classes are obviously "large", thus ruled out by any such "limitation of size" principle. In this respect, the situation is the same as in ZF set theory, where such "large" equivalence classes turn out to be "proper classes", not sets.

Having said that, the comparison to ZF set theory suggests an immediate response: Why not, within an updated Fregean theory of classes, use the construction of the natural numbers that goes back to von Neumann? The situation is as follows: Within the updated theory, Frege's construction of classes of equinumerous classes does not lead us to genuine objects, but at most to "quasi-objects", like proper classes. Yet we need numbers to be genuine objects (things that can themselves be elements of classes etc.) for purposes of higher arithmetic. The solution is not to use the equivalence classes themselves, but representatives from each of them instead, just like in ZF set theory. These representatives will themselves be unproblematic objects, as "small" classes. Moreover, if we use the particular representatives introduced by von Neumann, i.e., $0 = \varnothing$, $1 = \{0\}$, $2 = \{0, 1\}$, etc., the following can be observed: These representatives correspond naturally to Frege's original construction, in the sense that first-order concepts of which they are the extensions, namely "$x \neq x$", "$x = 0$", "$x = 0 \vee x = 1$", etc., are used in his construction of the equivalence classes.[39]

How might Frege have responded to such a proposal? This is, of course, a very speculative question and hard to answer; but it points towards another, potentially more tractable question. Note here, first, that within contemporary mathematics appeals to equivalence relations and uses of equivalence class constructions are very common, e.g., the construction of the system of integers modulo n in algebra. It is well known, moreover, that in such contexts one can often work either with the equivalence classes or with corresponding representatives.[40] Now, the use of representatives in the case of the integers modulo n, say, goes as far back as the early nineteenth century (Gauss and his successors). This makes it likely, or at least possible, that it was not unfamiliar to Frege.[41] Nevertheless, he did not adopt this technique for the natural numbers, not even after having been informed of Russell's antinomy. The question is: why not?

A superficial answer to that question is that, even had Frege tried to work with representatives instead of equivalence classes, his underlying theory of classes would still have been inconsistent, and he did not see a way of fixing that theory. (After all, the "limitation of size" idea only became prominent later, and it was only proposed very recently as a remedy for Frege's theory of classes.) In addition, perhaps we loose something important, from Frege's philosophical perspective, if we replace the original equivalence classes by representatives. Note, in particular, that the ordinary applications of arithmetic are then no longer built into the definition of the natural numbers, or at least not as directly. While sufficient for inner-mathematical purposes, the proposed modification might thus lack a feature important to Frege for other reasons.[42]

If this last suggestion is not completely off the mark, then the only modification of his system that would satisfy Frege in the end was one that, while preserving consistency, allowed for the full equivalence class construction of the natural numbers. But is such a modification possible at all? Perhaps the proposals discussed so far are all we can hope for. Support for the latter view comes from two sides: First, it may be argued that, in the context of a theory of classes or sets, the "limitation of size" idea gets at something

essential with respect to avoiding Russell's and similar antinomies. If this is the case, then the equivalence class construction is ruled out not just for a superficial, but for a deep reason. Second, also from a predicative perspective—the main alternative to set theory for avoiding the antinomies, as is often assumed—Frege's equivalence classes appear deeply problematic. In line with both points of view, a third observation can be added: Unlike in the case of other equivalence class construction, such as that for the integers modulo n, there is a kind of circularity, or non-well-foundedness, built into Frege's construction. Namely, numbers introduced as equivalence classes do contain elements that again contain the same numbers. The underlying phenomenon here is this: We do not only want to number other things, but also numbers themselves (e.g., in saying that there are four prime numbers between 1 and 10); but then, classes containing numbers will be elements in Frege's equivalence classes. Isn't that problematic, indeed obviously and irrevocably so?

I want to conclude my discussion of possible neo-Fregean modifications of his original theory by challenging this line of thought, thus also pointing towards yet another neo-Fregean possibility. The challenge is this: Might it not be possible, in spite of such arguments, to restrict Frege's theory of classes in such a way that it not only turns out to be consistent, but still allows for his full equivalence class construction? In fact, exactly such a modification was proposed already several decades ago: W. V. Quine's "New Foundations" (NF). The guiding idea in Quine's approach is to restrict the formation of classes not by excluding "large" ones, nor by excluding impredicative ones, but by only allowing defining clauses that respect certain syntactic structures (partly motivated by, but different from type-theoretic strictures). Crucially for present purposes, these syntactic strictures do not rule out the formation of Fregean equivalence classes. Actually, Quine noted this himself, and saw it as an advantage of his approach.[43]

A theory such as Quine's NF does not coincide entirely with what Frege tried to do in his initial reaction to Russell's antinomy. The remaining difference is that Frege's initial suggestion (his attempted, but failed "way out") would have allowed for extensions of concepts for all concepts, by way of weakening the logic of extensions in certain ways. Quine's suggestion, like those based on the "limitation of size" idea and like predicative proposals, is not as permissive, but only allows for extensions of concepts satisfying some additional condition. Still, Quine's update of Frege's theory may be the most "Fregean" of them all, at least if one accepts that Frege's original equivalence class construction is central and should not be given up, if at all possible.

Unfortunately, Quine's NF is not without its own problems. In particular, it is still not known whether it is consistent or not (relative to set theory). Indeed, the theory seems to be rather intractable in that respect. The conception of classes presented in it is also often considered to be "unintuitive", or at least less intuitive than the cumulative conception of sets that underlies ZF set theory. Then again, NF is not known to be inconsistent, in spite of allowing for Frege's equivalence class construction, and it has some other attractions.[44] In addition, even if one does not find Quine's particular proposal attractive, the fact that the availability of the Frege-Russell conception within it does not lead to an immediate, obvious inconsistency suggests a more general idea: Perhaps Frege need not have given up hope with respect to his project after all, even including the equivalence class construction for the natural numbers within a general theory of classes. More specifically, perhaps the problems he encountered do not have to do with that construction, but simply with the idea that every concept determines an extension. It seems possible, in other words, that the Frege-Russell construction can be completely separated from that problematic idea and saved, in Quine's or some other, more attractive theory of classes. At the very least, it appears that we still do not understand completely its connection to antinomies such as Russell's, if they are necessarily connected at all.[45]

Conclusion

In this paper I have reexamined Frege's conception of the natural numbers, especially with respect to its motivations and possible modifications. In terms of motivations, I have argued that this conception should be seen as growing out of the earlier pluralities conception of numbers, which shares with it the focus on ordinary applications of arithmetic. I have also discussed the basic moves Frege makes in improving on, and going beyond, the pluralities conception, including the individual motivations of these moves. In terms of modifications, I have surveyed both those that can be found in Frege's own writings, before and after he found out about Russell's antinomy, and several more recent neo-Fregean proposals. It is, again, very speculative to ask, and perhaps impossible to answer, which of those recent proposals would have appealed the most to Frege had he been confronted with them. Nevertheless, it is possible to observe a number of respects in which they are more or less "Fregean", as I have also done.

My discussion of the various proposals for rescuing Frege's system have been brief and sketchy, probably also one-sided in some respects. However, I hope that the following four general points have become evident along the way: First, it is clear now that Frege's theory, or large parts of it, can be saved from contradiction if one is willing to make certain modifications. Second, the various modifications that have been proposed have different advantages and disadvantages, especially relative to Frege's original goals. Third, a reflection back on the motivations and development of Frege's own views can shed light on these advantages and disadvantages. Fourth, even Frege's full equivalence class construction may possibly be resurrected

in a general theory of classes, although that possibility has still not been explored enough, and with it the precise implications of Russell's antinomy. My final conclusion is this: Frege would be very pleased to see how far from a "complete failure" his efforts were, after all, and how much fruitful research into logic and the foundations of mathematics they have inspired, especially recently.[46]

Notes

1 In the present paper I expand on work done in (Reck 2004). Part 1 (§1.1 and §1.2) contains summary treatments of issues dealt with already, in more detail, in the earlier paper; parts 2 and 3 go beyond it.

2 In what follows I will put aside the variant of the pluralities conception according to which numbers are multitudes of "pure units", where the notion of "unit" is understood in an abstract way, and in such a way that a numeral turns out to refer to be a singular term after all (e.g., "2" refers to the unique multitude consisting of two such units). For more on that variant, including Frege's criticism of it, see (Reck 2004).

3 Frege himself sometimes uses the phrase "kindergarten-numbers [Kleinkinder-Zahlen]" in this connection; see (Frege 1924/1925b). (I will come back to that late note of his below, in §2.2.)

4 In this section (unlike in §2.1 below), I am not so much guided by the chronological order in which Frege introduced his ideas but by the conceptual order emerging from the discussion in §1.1.

5 Here I follow (Wilson 1992), especially with respect to the role of geometry for Frege (although the actual account suggested in Wilson's paper is richer and more complex than indicated); compare also (Tappenden 1995). It would be interesting to know to what degree, and in what form exactly, Frege was also aware of equivalence class constructions in the algebra of his time, e.g., what today would be called the constructing of cosets of natural numbers modulo *n*. (I will come back to this issue in §3.2.)

6 According to Frege's views about the reference of "the concept *F*", the phrase "the concept 'two-ness'" should perhaps be taken to refer to the corresponding equivalence class anyway. This is what could be behind the cryptic footnote on p. 80 of (Frege 1884), as suggested in (Burge 1984) and (Ruffino 2003). But compare (Wilson 2005) for a different interpretation. (I will come back to this issue in §2.1.)

7 Compare the corresponding quotations and references in (Reck 2003).

8 As pointed out to me by Gottfried Gabriel, it is possible, or even likely, that before *Begriffsschrift* Frege conceived of the natural numbers as second-order numerical concepts. Much later he came back, very tentatively, to this idea, including the possibility of ordering these concepts in a sequence; see (Frege 1919). (I will come back to this issue in §2.2 below.)

9 Occasionally, as here, I have amended J. L. Austin's standard translation of Frege 1884 slightly.

10 There is evidence that Frege took over the analysis of numerical statements as statements about concepts from writers he read as a student, in particular Johann Friedrich Herbart; see (Gabriel 2001). However, the full force of that analysis only becomes apparent when combined with two distinctively Fregean moves: placing it within the framework of his new logic; sharply distinguishing between concepts and objects.

11 Compare here n. 6.
12 The sense-reference distinction is usually discussed in its application to object names in the literature, as Frege does himself in "Über Sinn und Bedeutung". I take the application to function and concept names to be another important motivation for its introduction, however, as made more explicit by Frege in "Über Begriff und Gegenstand" and in "Ausführungen über Sinn und Bedeutung" (Frege 1983, pp. 128–136).
13 Compare the analysis of Frege's *Grundgesetze* construction in (Quine 1954), p. 149.
14 Actually, even in *Grundgesetze* Frege expresses a slight hesitation about classes; or at least he points towards Basic Law V as a possible weak point of his system (Frege 1893, p. VII). (Compare §2.2 below.)
15 Frege makes several of these points also in another letter to Russell, dated July 28, 1902; see (Frege 1980) especially pp. 140–141. (I will come back to this letter below, at the end of §2.2.)
16 See (Quine 1954) for a classic discussion of the reasons why "Frege's way out" fails.
17 See Appendix B to "Begriffsschrift I" in (Frege 2004).
18 See (Wittgenstein 1921), 6.021 etc. For more on Frege's relation to Wittgenstein, including their conversations and correspondence during this period, see (Reck 2002).
19 Compare (Ruffino 2003), in which the special status of extensions in Frege's logic is defended further.
20 Compare (MacFarlane 2002) in this connection. (More on this issue below, at the end of §3.1.)
21 Gottfried Gabriel makes the former point in the introduction to (Frege 1996), in the context of a brief review of the development of Frege's views about numbers (on which I have drawn in the present paper).
22 Compare the discussion of Frege's views about the complex numbers in (Simons 1995).
23 See (Bostock 1974), (Hodes 1984), and the discussion of "numerical quantifiers" in them. In some recent, still unpublished work by Aldo Antonelli and Robert May related ideas are being explored.
24 In line with (Tait 1997), (Ricketts 1997), and (Demopoulos 1998), I think that the choice of the name "Hume's Principle" is unfortunate. "Cantor's Principle" would have been more justified, and the neutral "the contextual definition of numbers" perhaps best. But not much hangs on this terminology for my purposes.
25 See (Wright 1983) and (Hale and Wright 2001), the relevant articles in (Demopoulos 1995) and (Boolos 1998), and the general discussions in (Fine 2002) and (Burgess 2005).
26 See the summary of *Grundgesetze* in (Heck 1993), as well as the discussion in (Demopoulos 1998).
27 This is sometimes presented as a crucial advantage of the approach; see the discussion, and defense, of "Frege's constraint" in (Wright 2000) and in the introduction to (Hale and Wright 2001).
28 See again (Frege 1902b), pp. 140–141. Here, I follow (MacFarlane 2002), especially the section entitled "Generality and Hume's Principle".
29 This last reason is suggested in (Ricketts 1997), p. 196.
30 As his remarks on "definitions by abstraction" in (Frege 1902b), p. 141, indicate, Frege himself was aware of this historical connection, via Russell. Here I am indebted to Michael Beaney.
31 Wright and Hale defend their use of abstraction principles via Frege's context principle, thus giving their approach a more "Fregean" appearance again, although the point is not uncontroversial.

32 This observation, like the previous two, is not meant as an argument against the Wright-Hale conception itself. I hope to be able to explore the compatibility of structuralist views and this kind of neo-logicist view further in a future publication. See (Reck 2003) for what I take to be Dedekind's logical structuralism.
33 See (Russell 1908) and (Whitehead and Russell 1910).
34 Compare, e.g. (Ferreira and Wehmeier 2002) and (Linnebo 2004). (Burgess 2005), Chapter 2, contains a systematic discussion of such approaches, including further references.
35 See (Dummett 1991), pp. 226*ff.*; but compare the critical discussion in (Wright 1998).
36 Compare here the criticisms of Russell's axioms of reducibility, infinity, etc. in (Wittgenstein 1921). See also the Frege-Russell correspondence, especially (Russell 1902b) and (Frege 1902c), and Frege's rejection of early Russellian suggestions to view numbers as "improper objects" in it.
37 Compare (Boolos 1986/1987), (Hale 2000), and Chapter 3 in (Burgess 2005).
38 See again Chapter 3 (Burgess 2005) for a discussion of Friedman's proposal.
39 In (Boolos 1987), pp. 227–229, the author goes so far as calling these extensions "the true Frege finite cardinals"; compare the discussion in (Demopoulos 1998), pp. 484–486.
40 See (MacLane and Birkhoff 1993), Chapter I, sections 8–9, for a classic presentation.
41 I am not aware of any historical account of the use of such methods in nineteenth-century mathematics, or even earlier; thus I am not sure how safe it is, in the end, to assume that Frege knew about them. It would be interesting to explore this issue further, but I cannot do so here. (Compare here n. 5.)
42 Compare here again (Demopoulos 1998), especially section IV.
43 See (Quine 1969) for a general introduction to New Foundations, also (Rosser 1953) for a detailed discussion of the Frege-Russell construction within this framework.
44 See (Forster 1995) for a relatively recent, systematic discussion of NF and related approaches. For an earlier summary of problems and questions concerning NF, compare (Wang 1986).
45 Compare here (Boolos 1987), in which the (relative) consistency of simply adding an axiom asserting the existence of Fregean equivalence classes to second-order logic is established. However, the possibility of a more general theory of classes (such as Quine's NF) in which this existence claim becomes a theorem is not explored in that paper. The latter is what is at issue in the present discussion.
46 Earlier versions of this paper were presented as talks at the University of California at Berkeley (October 2003) and at the University of California at Irvine (May 2004). I am grateful to Paolo Mancosu and Kai Wehmeier for the respective invitations, and to members of the two audiences for valuable feedback. I would also like to thank Michael Beaney, Martin Davis, William Demopoulos, Gottfried Gabriel, John MacFarlane, Thomas Ricketts, Kai Wehmeier, and Joan Weiner for helpful comments on later drafts. As usual, all responsibility for the remaining mistakes lies with me.

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