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DEDEKIND'S STRUCTURALISM: AN INTERPRETATION AND PARTIAL DEFENSE

ABSTRACT. Various contributors to recent philosophy of mathematics have taken Richard Dedekind to be the founder of structuralism in mathematics. In this paper I examine whether Dedekind did, in fact, hold structuralist views and, insofar as that is the case, how they relate to the main contemporary variants. In addition, I argue that his writings contain philosophical insights that are worth reexamining and reviving. The discussion focusses on Dedekind's classic essay "Was sind und was sollen die Zahlen?", supplemented by evidence from "Stetigkeit und irrationale Zahlen", his scientific correspondence, and his Nachlaß.

1. INTRODUCTION

Structuralist views, or a structuralist approach, have become a major theme in recent philosophy of mathematics. Two of the main proponents of such views are Geoffrey Hellman and Stewart Shapiro. Both Hellman and Shapiro present, or try to appropriate, the mathematician Richard Dedekind (1831–1916) as the distinguished forefather of their respective positions. Hellman starts *Mathematics without Numbers*, his main work on the topic, as follows:

> The idea that mathematics is concerned principally with the investigation of structures of various types in complete abstraction from the nature of individual objects making up those structures is not a novel one, and can be traced at least as far back as Dedekind's classic essay, "Was sind und was sollen die Zahlen?" (Hellman 1989, p. vii).

He goes on to present in detail what he calls "modal structuralism", his preferred variant of structuralism. Stewart Shapiro, in a number of recent writings, has defended a view he calls "*ante rem* structuralism". In *Philosophy of Mathematics: Structure and Ontology*, the main presentation of his position, he writes:

> A direct forerunner of *ante rem* structuralism is [...] Dedekind. His development of the notion of continuity and the real numbers, in ["Stetigkeit und irrationale Zahlen"], his presentation of the natural numbers via the notion of Dedekind infinity, in ["Was sind und was sollen die Zahlen?"], and some of his correspondence constitute a structuralist manifesto. (Shapiro 1997, p. 14)

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However, neither Hellman nor Shapiro is all that interested in exegetical questions, so that what they say about Dedekind remains sketchy and general. Their real interest is in arguing for their own respective views.

In this paper I want to focus on Dedekind's works, in particular "Was sind und was sollen die Zahlen?", his classic essay on the natural numbers. My goal will be to give a careful, historically informed, and philosophically nuanced interpretation of Dedekind's basic position in this essay, with the focus on logical, semantic, and metaphysical aspects of it. My main concern will be to answer the question of whether this position is indeed "structuralist", and if so, what exactly that means. This will involve comparing it to several of the most prominent variants of structuralism in the current literature, including those by Hellman and Shapiro. I will conclude that, in spite of some connections, Dedekind's position is different from theirs in important ways. Thus their appropriations of Dedekind have to be taken with a grain of salt.

To give a definite interpretation of Dedekind's position in comparison with current structuralist views is not an easy task, for two reasons: First, Dedekind leaves us with only a few brief, but pregnant philosophical remarks scattered throughout "Was sind und was sollen die Zahlen?". In order to understand their significance it will be important, then, to supplement them with material from several other sources: from his other published writings, from his scientific correspondence, and from his unpublished Nachlaß. It will also help to consider the relation of certain of Dedekind's core ideas to those of his contemporaries, or near-contemporaries, Dirichlet, Riemann, Cantor, and Hilbert. Second, it is not the case that there is only one contemporary position, "structuralism", with which to compare Dedekind's position. Rather, recent structuralist approaches differ in a number of crucial respects, as the distinction between "modal" and "*ante rem* structuralism" above already indicates. For that reason it will be necessary to start with a brief overview and comparison of the main current structuralist positions.[1]

2. CONTEMPORARY STRUCTURALIST POSITIONS

While there are important differences between contemporary structuralist positions, there is also some overlap, or at least a common starting point. This starting point consists in the vague, but suggestive thesis that, to use Hellman's words again, "mathematics is concerned principally with the investigation of structures [...] in complete abstraction from the nature of individual objects making up those structures" (Hellman 1989, p. vii). Similarly Shapiro writes: "[P]ure mathematics is the study of structures, independently of whether they are exemplified in the physical realm, or in any realm for that matter" (Shapiro 1997, p. 75). What is central, then, is the notion of "structure", together with the related notion of "abstraction".

The differences between current structuralist positions become evident when the identity and nature of such "structures", as well as the precise meaning of "abstraction" in this connection, are probed further. In terms of a general taxonomy, let me distinguish four main alternatives, i.e., four different responses to such probing, which all play a role in the current discussions, explicitly or implicitly. To be able to refer to them briefly and succinctly, I will call the corresponding positions "methodological structuralism", "set-theoretic structuralism", "modal structuralism" and "*ante rem* structuralism".[2]

As the term "methodological structuralism" suggests, this first position has primarily to do with mathematical *method*, rather than with semantic and metaphysical issues as the others do. Thus it is really in a separate category, or of a different kind. Nevertheless, it will be helpful to introduce it alongside the others, for contrast. In order to understand what methodological structuralism amounts to, considering the example of modern abstract algebra, including the theory of groups, rings, moduls, fields, etc. What modern algebraists do is to study various *systems of objects*, of both mathematical and physical natures (the latter at least indirectly), which satisfy certain general conditions: the defining axioms for groups, rings, moduls, fields, etc. More precisely, they study such systems *as* satisfying these conditions, i.e., as groups, rings, etc. Also applicable beyond algebra to arithmetic, geometry, etc., *methodological structuralism* consists then of such a general, largely conceptual approach (as opposed to more computational and particularist approaches). It is typically tied to presenting mathematics in a formal axiomatic way.[3]

As just described, a mathematician who is a methodological structuralist will not be concerned about the further identity or nature of the objects in the various systems studied. He or she will simply say: Wherever they come from, whatever their identities and natures, in particular whatever further "non-structural" properties these objects may have, insofar as a system containing them satisfies the axioms ..., the following is true of it: ... . This is the sense in which methodological structuralism involves a kind of *abstraction*. Here abstraction concerns simply the question which aspects of a given system are studied and which are ignored when working along such lines.

If pressed further, especially if asked for a general, systematic answer to the question of what the range of the systems studied consists in and how to conceive of them, a methodological structuralist will typically point to set theory. That is to say, set theory, usually as based on the Zermelo-Fraenkel axioms, is presented as the right framework in which to answer such questions, in particular to answer them mathematically. From a philosophical point of view this leads to a new question, though: What does such an appeal to set theory imply about the content of mathematics, or about the identity and nature of mathematical objects? For everyday mathematical purposes that question can be put aside – answering it is not part of methodological structuralism. But if not simply dismissed, it leads naturally to our second main position: *set-theoretic structuralism.*

To understand this second position, let us focus on arithmetic as a typical and relatively simple example. What Zermelo-Fraenkel set theory allows us to do is to construct *models* of the corresponding axioms: the Dedekind-Peano Axioms. Indeed, it allows us to construct various such models, various "natural number systems", all build up of sets (pure sets, usually). A natural number system consists, then, of a triple of things: an infinite set, usually the finite von Neumann ordinals ω; a distinguished element in that set, usually the empty set $\emptyset$; and a successor function defined on the set, usually $suc : x \mapsto x \cup \{x\}$. Similarly for the real numbers, and one can construct examples of other groups, rings, fields, etc. in set theory as well. Crucially, such set-theoretic models can now be seen as the "systems" we talked about earlier, i.e., the objects of study for a methodological structuralist. Actually, the term most often used for such systems in the corresponding literature is "structures".

Simply using set theory as the framework for most or all of mathematics, as just described, does not yet amount to "set-theoretic structuralism" in my sense of the term. What has to be added is the commitment to certain *semantic* and *metaphysical theses*. The central metaphysical thesis is that all there exists, or at least all we need to assume to exist, in mathematics is sets. The central semantic thesis is that when we talk about "the natural numbers", "the real numbers", etc., we are talking about sets, along the model-theoretic lines indicated above. Now, if we ask which particular sets the natural numbers, say, consist of, another crucial aspect of set-theoretic structuralism comes to the fore.

Namely, a set-theoretic structuralist maintains (often not explicitly, but implicitly) that the answer to this last question is *relative*, not absolute. That is to say, it depends on a largely arbitrary *choice* of one model for the Dedekind-Peano Axioms within set theory. The usual – only pragmatically and weakly justified – choice is the finite von Neumann ordinals. But various other choices are possible as well, e.g., the finite Zermelo ordinals or some permutation of them. And crucially, all such choices are *equivalent.* It is the emphasis on this last point that makes the position in question

again "structuralist", since it amounts to the claim that all that matters are "structural" facts, i.e., those facts that are invariant under such choices of model. In other words, by considering all the models to be equivalent we *abstract away* from everything else.

To repeat, when a set-theoretic structuralist talks about "structures" he or she has in mind set-theoretic objects, e.g., certain triples of pure sets satisfying the Dedekind-Peano Axioms. For an *ante rem structuralist* such as Steward Shapiro the identity and nature of structures is conceived of quite differently. For him, the "natural number structure" is not identical with any of the set-theoretic models of the axioms. Rather, it is what they all have in common, or what they all "instantiate" and "exemplify". That is to say, the structure in question is now thought of as a *universal.* As such it is distinct from all the set-theoretic systems which are *particulars* (at least as usually conceived).

The difference at issue here can be explained further as follows: All the particular natural number systems introduced earlier consist of sets, which in turn have other sets as elements (except for the empty set). In contrast, the new natural number structure is not a set, nor are its "components" sets. Instead, these components, or parts, are completely structureless "points" or "places", to be filled by particular objects in an exemplification. Similarly for, say, the real number structure. (It is crucial at this point that we are dealing with categorical axiom systems.) Thus, the universal structures of an *ante rem* structuralist are conceived of as different in their *nature* from all particular systems of objects which exemplify them. They are also considered to be independent in their *existence* from them – this is what it means for structures to be *ante rem.* Finally, if we ask about which such structures exist, the answer is given in a separate *structure theory*, parallel to Zermelo-Fraenkel set theory.

An *ante rem* structuralist is, thus, willing to postulate additional "abstract" entities, even beyond sets (which are already abstract in some sense). The main motivation behind *modal structuralism*, as developed by Geoffrey Hellman, is precisely to avoid the postulation of all abstract entities, including *ante rem* structures and sets. That is to say, modal structuralism, our fourth contemporary alternative, is intended to be an *eliminativist* view. But how is this elimination of abstract entities supposed to be accomplished? The crucial step is this: We are urged to interpret, or analyze, all mathematical sentences in a new way.

We can take a simple arithmetic sentence as an example: '2 + 3 = 5'. For a modal structuralist, with this sentence we are not talking about sets; nor are we talking about an additional abstract entity, an *ante rem* structure. Rather, the sentence is to be analyzed as a *universally quantified* sentence, of the following general form: For all models of the Dedekind-Peano axioms, if we add the 2-element to the 3-element in it, then we get its 5-element as a result. Note that, along these lines, we again *abstract away* from the nature of particular objects in these models – this time by quantifying them out. Actually, there is one more ingredient we need to add to this analysis: a modal operator. In the end, what '2 + 3 = 5' really says, according to a *modal* structuralist, is that the universally quantified statement just mentioned is *necessarily* the case. Similarly for all other mathematical sentences.[4]

To sum up our discussion in this section: A set-theoretic structuralist works with "structures" in the sense of familiar set-theoretic systems or models, and does so in a relativist way. An *ante rem* structuralist postulates additional abstract entities apart from them, namely "structures" in the sense of universals, in themselves consisting of structureless points or places. A modal structuralist, in turn, avoids the postulation of all such abstract entities, by re-analyzing in a universal-modal way what mathematical sentences say. All three of these positions involve a kind of "abstraction"; but what this abstraction amounts to differs significantly from case to case. Finally, all three can be seen to be guided by, or at least compatible with, methodological structuralism, which, as a position in itself, just has to do with mathematical methodology, not with semantics and metaphysics.

3. *WAS SIND UND WAS SOLLEN DIE ZAHLEN?*

Let us turn to Richard Dedekind now. So far I have distinguished four main contemporary variants of structuralism. On the basis of this taxonomy we can ask: Was Dedekind a structuralist? And if so, was he a structuralist in any of these four senses? In addressing these questions, my main source of evidence will be "Was sind und was sollen die Zahlen?" (Dedekind 1888b), Dedekind's classic essay on the natural numbers, but occasionally it will prove useful to go beyond it.[5]

Considered in general, Dedekind's works are among the first in the history of mathematics to employ set-theoretic notions and techniques in a serious, systematic way. For present purposes it is important to add, however, that there are also some interesting differences to contemporary set theory. Dedekind's basic framework, in "Was sind und was sollen die Zahlen?" and elsewhere, is this: He works with a naive theory of functions and sets – he calls them "systems" – in the background. This background theory is "naive" in two senses: it is not axiomatized; it allows for arbitrary collections of objects as sets, i.e., Dedekind implicitly accepts an unrestricted comprehension principle. A further difference to current set theory, even current naive set theory, is that functions are not reduced to sets by him, but are taken to be primitive as well.

In "Was sind und was sollen die Zahlen?" this machinery is used to carry out several ingenious set-theoretic constructions. In particular, given a set S on which a 1-1 function ϕ is defined, Dedekind considers the *chain* formed by the function over a given element a in the set, i.e., the smallest subset that contains a and is closed under ϕ. In terms of terminology, Dedekind calls a 1-1 function "similar", and he uses the notation a_0 for such a chain over the element a. He also defines a set S to be *infinite* if it "is similar to", i.e. can be mapped 1-1 onto, a proper subset of itself.[6]

Next Dedekind introduces the central notion of a *simple infinity*:

> 71. Definition. A system N is said to be *simply infinite* when there exists a similar function ϕ of N in itself such that N appears as chain (44) of an element not contained in $\phi(N)$. We call this element, which we shall denote in what follows by the symbol 1, the *base-element* of N and say that the simply infinite system N is *set in order* by this function ϕ. (Dedekind 1963, p. 67, original emphasis.)

Note that, by using the symbols '1' and 'N', Dedekind is already indicating that simply infinite systems have something to do with the set of natural numbers. Later in the same Definition 71 he adds:

> [T]he essence of a simply infinite system N consists in the existence of a function ϕ on N and an element 1 which satisfy the following conditions $\alpha, \beta, \gamma, \delta$:
> α. $\phi(N) \subset N$.
> β. $N = 1_0$.
> γ. The element 1 is not contained in $\phi(N)$.
> δ. The function ϕ is similar. (*Ibid.*)[7]

It is not hard to see that this list, or this overall "condition" on a set N, amounts to Dedekind's formulation of the *Peano Axioms*; or better, the *Dedekind-Peano Axioms*, as Peano acknowledged his indebtedness to Dedekind's work.[8] In addition, using his definitions Dedekind can show easily (Theorem 72) that any infinite system S contains a simply infinite system N as a subset.

To a contemporary set theorist the definitions and constructions so far will look very familiar. In Dedekind's time, however, they were brand new and highly original. Their current familiarity is simply a reflection of the depth of Dedekind's influence. Having said that, if we move one step further in Dedekind's essay a striking difference emerges. In contemporary set theory the existence of an infinite set, thus also that of a simply infinite set (an "ω-sequence"), is guaranteed axiomatically, via the *axiom of infinity*. Dedekind, in contrast, does not use such an axiom; instead he thinks he can provide a *proof* that infinite sets exist (Theorem 66). Infamously, this proof starts with "the totality S of all things which can be objects of my thought". From there, it proceeds by picking one particular element a in S so as to establish the following: The chain over a formed by the function $f : x \mapsto$ "x can be the object of my thoughts" is contained in S and, as a subset, forms a simple infinity (*ibid.*, p. 64).

Nowadays this "proof" is considered to be fundamentally problematic, for at least two reasons: First, it involves the notion of an "object of thought", thus seemingly introducing psychologistic considerations into mathematics. Second and more clearly, it involves the notion of a "totality of all things (which can be objects of my thought)", which leads directly to Russell's antinomy, as is not hard to show. In fact, after finding out about that antinomy later on, it was the reason why Dedekind himself came to see the proof as fundamentally problematic.[9]

I want to postpone discussions of psychologistic, or seemingly psychologistic, elements in Dedekind's views until later. I also want to put aside questions about consistency, as raised by Russell's antinomy and related results, for a while.[10] For the moment, let us just assume that an infinite system exists, which implies than one, indeed many, simply infinite systems exist as well. Where does Dedekind go from there in his essay?

Shortly after his definition of a simply infinite system (Definition 71), we can find the following central and often quoted remark by Dedekind:

73. Definition. If in the consideration of a simply infinite system N set in order by a function ϕ we entirely neglect the special character of the elements, merely retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting function ϕ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. (*Ibid.*, p. 68, original emphasis.)

This passage is the place where "the natural numbers", the central topic of Dedekind's essay, are introduced by him. Explaining their introduction further he continues:

With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind. The relations or laws which are derived entirely from the conditions $\alpha, \beta, \gamma, \delta$ in (71), and which are therefore always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements (compare 134), form the first object of the *science of numbers* or *arithmetic*. (*Ibid.*, original emphasis)

For later considerations, note in these two (continuous) passages especially the following phrases: "neglecting the special character of the elements", "freeing the elements from every other content", and "free creation of the human mind". What Dedekind expresses in terms of them is all part of the "abstraction" that is crucial to his conception of the natural numbers. Note

also that according to him it is everything that can be "derived" from his four conditions, or axioms, that constitutes what arithmetic is all about.

Clearly interpretations of the passages just quoted, cryptic as they are, will be central for understanding Dedekind's conception of the natural numbers. Before attempting such interpretations, I want to bring into play two other parts of "Was sind und was sollen die Zahlen?" that are closely related. The first consists of two theorems that are proved a number of pages further on in the essay:

132. Theorem. All simply infinite systems are similar to the number-series N and consequently (33) also to one another.[11]

133. Theorem. Every system that is similar to a simply infinite system and therefore (132), (33) to the number-series N is simply infinite.

(*Ibid.*, pp. 92–93.)

While Dedekind is not completely precise, or at least not completely explicit, about the notion of isomorphism involved here – he defines "similarity" in terms of the existence of a 1-1 ("similar") function, while what he really needs is a bijective function (1-1 and "onto") – his proofs of these two theorems show that he essentially understands the notion of a simple infinity, thus the Dedekind-Peano Axioms, to be *categorical*.[12]

The second additional part of Dedekind's essay that needs to be brought into play is a remark following these two theorems directly:

134. Remark. By the preceding theorems (132), (133) all simply infinite systems form a class in the sense of (34) [an equivalence class under "similarity"]. At the same time, with reference to (71), (73) it is clear that every theorem regarding numbers, i.e., regarding the elements n of the simply infinite system N set in order by the function ϕ, and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the order-setting function ϕ, possesses perfectly general validity for every other simply infinite system Ω set in order by a function θ and its elements ν; and that the transition from N to Ω (e.g., also the translation of an arithmetic theorem from one language into another) is effected by the function ψ considered in (132), (133), which maps every element n of N to an element ν of Ω, namely to $\psi(n)$. This element ν can be called the nth element of Ω, and accordingly the number n is itself the nth number of the number series N. The same significance that the function ϕ possesses for the laws in the domain N, insofar as every element n is followed by a determinate element $\phi(n) = n'$, is found, after the transition effected by ψ, to belong to the function θ in the domain Ω, insofar as the element $\nu = \psi(n)$ as the image of n is followed by the element $\theta(\nu) = \psi(n')$ as the image of n'. [...] By these remarks, as I believe, the definition of the notion of number given in (73) is fully justified. (*Ibid.*, pp. 95–96.)

What Dedekind observes in this passage is basically the following: that the *categoricity* of his notion of "simple infinity" implies the *logical equivalence* of all corresponding "models", in the sense that all of these models satisfy exactly the same arithmetic sentences. Moreover, this is taken to "justify" his introduction of the natural numbers in Definition 73.

Having said that, note also that Dedekind does not call his four conditions $\alpha, \beta, \gamma, \delta$ "axioms". Nor does he, strictly speaking, think in terms of models of an uninterpreted, formal language. Rather, he presents his conditions as the definition of a (higher-level) *concept*: that of a simply infinite system. And he writes about "translating" between different languages used to talk about various systems falling under that concept. In those respects his approach is different from the usual contemporary one. Put briefly, Dedekind's approach is more *conceptual* than formal or model-theoretic.

A final observation about "Was sind und was sollen die Zahlen?": After proving Theorems 132 and 133, also after making Remark 134, Dedekind concludes his essay by providing, in a series of theorems, the following: a careful justification of proofs by mathematical induction, and of corresponding inductive definitions; a definition of the order relation, then of the operations of addition, multiplication, and exponentiation for the natural numbers; and an explanation of the use of initial segments of the number series to measure the cardinality of finite sets. What he thus shows is that all the usual arithmetic results can be reconstructed along his lines.

4. STRUCTURALIST ASPECTS: FIRST COMPARISONS

How should we interpret these Dedekindian definitions, theorems, and remarks? Are they the expression of one of the structuralist positions described above; and if so, which one? Or do they constitute a different position, structuralist or otherwise?

It is obvious, I think, that Dedekind's "Was sind und was sollen die Zahlen?" embodies a *methodological structuralist* approach to the natural numbers. That is to say, he clearly studies the system of natural numbers as a structure, i.e., with respect to its structural properties. This is illustrated not only by his basic definition of a simply infinite system, but also by his subsequent theorems and proofs, especially the categoricity result. And if we go beyond this essay to Dedekind's other writings, we can see the same basic approach applied more widely: in Dedekind's introduction of the order-completeness property to characterize the real numbers; in his theory of ideals in algebraic number theory; in his early appreciation of Galois theory; and in his introduction of various basic algebraic notions such as that of a ring. Finally, there is the strong influence Dedekind had on subsequent generations of methodological structuralists, from Hilbert, Noether, and van der Waerden, to Bourbaki and others. No doubt, then: Dedekind was a methodological structuralist.[13]

But what about our three more philosophical versions of structuralism: set-theoretic structuralism, *ante rem* structuralism, and modal structuralism? On a first glance, it may look like Dedekind's remarks point in all three of these directions, thus inviting the respective appropriation by philosophers such as Hellman and Shapiro. Among the three, the case of set-theoretic structuralism is probably the most tempting. Thus I will consider it first in the present section, together with a few remarks about the less tempting case of modal structuralism. I reserve comparisons of Dedekind's views to *ante rem* structuralism for later.[14]

As already noted, Dedekind uses many techniques familiar from contemporary set theory in his works, including in "Was sind und was sollen die Zahlen?". He also clearly considers various different simple infinities, thus different models of the Dedekind-Peano Axioms, and compares them. Against that background, the passage that invites a *set-theoretic structuralist* reading the most is the following (already quoted more fully above):

> If in the consideration of a simply infinite system N, set in order by a function ϕ, we entirely *neglect* the special character of the elements, merely retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting function ϕ, then are *these elements* called *natural numbers*. (Dedekind 1963, p. 68, first two emphases added)

Remember that this passage, as part of Definition 73, occurs after Dedekind's "proof" of the existence of an infinite system S, and after his construction of a simply infinite system N as a subset of S. In other words, it occurs just after Dedekind has picked one particular simply infinite system N to work with. At this exact point he seems to be saying the following: we can treat *that very system* as "the natural numbers", provided that we "neglect" all aspects of it that go beyond it being a simply infinite system. Now, this sounds exactly like set-theoretic structuralism: construct some arbitrary simply infinite system by set-theoretic means, then treat *it* as the natural numbers, doesn't it?

There is a further detail in the text, easy to overlook in itself, that may be taken to provide additional support for such an interpretation. This detail has to do with Dedekind's notation. Notice that initially, in Definition 71, Dedekind uses the symbol 'N' to talk about simply infinite system in general. In Theorem 72 he then proves the existence of a particular simply infinite system to which he also refers by using 'N' (thus seemingly switching from using 'N' as a variable to using it as a constant symbol). After that, Dedekind *keeps using* that same symbol 'N'. In particular, he does *not* introduce a new symbol for "the natural numbers" as "created" in Definition 73. This suggests that he is still talking about the simple infinity constructed in Theorem 72, not some other, special simple infinity. He also keeps using N later on, e.g., in Definition 74:

From the general notion and theorems of IV. about the mapping of a system into itself we obtain immediately the following fundamental laws, where $a, b, \ldots, m, n, \ldots$ always denote elements of N, therefore numbers; $A, B, C, \ldots$ parts of N. (*Ibid.*, pp. 68-69)

To repeat, the symbol 'N' seems to refer to the same simply infinite system throughout Dedekind's essay, or at least from Theorem 72 on; likewise, '1', '2', '3', ..., and 'n', 'm', ... seems to refer to or range over the elements of that same simple infinity.[15]

Beyond issues of notation, Dedekind could, of course, have chosen any other simple infinity besides that constructed in Theorem 72, given his categoricity results in Theorems 132 and 133. To emphasize that fact – the *arbitrariness* of the choice of a model – is a second main ingredient in set-theoretic structuralism. And such an emphasis is exactly in line with Dedekind's Remark 134, especially (again) this part:

[W]ith reference to (71), (73) it is clear that every theorem regarding numbers, i.e., regarding the elements n of the simply infinite system N set in order by the function ϕ, and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the order-setting function ϕ, possesses perfectly general validity *for every other simply infinite system* Ω set in order by a function θ and its elements ν. (*Ibid.*, p. 95, emphasis added)

Thus, it doesn't really matter, especially with respect to arithmetic truth (or "validity", as Dedekind puts it), which simple infinity we pick.

The way in which set-theoretic structuralism, in our sense, goes beyond methodological structuralism is by adding certain specific philosophical theses to it. Once more, those theses concern the identity and nature of *structures*, on the one hand, and the corresponding notion of *abstraction*, on the other. To read Dedekind as a set-theoretic structuralist amounts, then, to two things: first, to interpret "abstraction" in the weak sense of "putting aside", or "not taking into account for present purposes", any additional non-structural properties – which seems exactly what is suggested in Dedekind's own phrases of "neglecting" and "leaving entirely out of consideration"; second, to consider only "structures" in the sense of set-theoretic systems – exactly like the one constructed by Dedekind in Theorem 72.

Strong evidence, then, to read Dedekind as a set-theoretic structuralist. In fact, it is hard to get away from such a reading if one is used to thinking in current set-theoretic and model-theoretic terms. Everything seems to line up so perfectly, and the resulting interpretation makes Dedekind

look both attractive and contemporary. However, we will soon encounter even stronger evidence against such an interpretation. This evidence will come from three sources: from other parts of "Was sind und was sollen die Zahlen?"; from Dedekind's concurrent scientific correspondence; and from an earlier version of his essay as contained in his Nachlaß. But before considering that evidence, let me turn to the claim that Dedekind's essay points also in the direction of *modal structuralism*.

Here we can be briefer, since with respect to this variant of structuralism it is much harder to make a case that Dedekind adhered to it. To begin with, I do not see any evidence that he considered modality as being crucial for our understanding of arithmetic statements. Even less do I see evidence that a re-interpretation, or re-analysis, of every arithmetic sentence along modal-structuralist lines is called for in his opinion. On the contrary, Dedekind can be seen to provide an account of arithmetic according to which we are to take arithmetic sentences "at face value" (more on this below). Furthermore, Dedekind has no scruples to accept "abstract" entities, including sets; his was, thus, not an eliminativist view (again, more on that below).

Nevertheless, there is one aspect of Dedekind's position that is at least compatible with modal structuralism, in particular with the *universalist* side of it. This aspects involves again the crucial structuralist insight in Remark 134: that every theorem regarding one simple infinity can be "transferred" to every other simple infinity. The way a modal-structuralist uses this insight is by pointing out that one can regard every such theorem as, implicitly, a statement about *all* simple infinities.

Interestingly, in an early comment on Dedekind's essay Bertrand Russell focusses exactly on this point. In his *Principles of Mathematics* (published in 1903) he writes:

What Dedekind presents to us is not the numbers, but *any* progression: what he says is true of *all* progressions alike, and his demonstrations nowhere – not even where he comes to cardinals – involve any property distinguishing numbers from other progressions. [...] Dedekind's ordinals are not essentially either ordinals or cardinals, but the members of *any* progression. (Russell 1903, pp. 249–251, emphasis added)

As this passage shows Russell, for one, interpreted Dedekind exactly as subscribing to the universalist side of modal structuralism (even if not the modal side). It is, however, hard to defend such an interpretation based on a close reading of Dedekind's text – it seems more a matter of Russell projecting his own "universalist" views about geometry onto Dedekind.[16]

The only additional way in which modal structuralism can be seen to be connected to Dedekind's position is this, it seems to me: Modal structuralism is, at least partly, guided by a methodological structuralist approach to mathematics; and Dedekind's work is certainly an early example of that, as observed above. Then again, the other philosophical variants of structuralism are also, partly or wholly, guided by methodological structuralism. Consequently this fact does not speak particularly in favor of a modal structuralist reading of Dedekind.

5. FREE CREATION: THE NATURAL AND THE REAL NUMBERS

So far an interpretation of Dedekind as a set-theoretic structuralist, in addition to being a methodological structuralist, looks most promising. Now I want to make several observations that weigh against such an interpretation. Considering the corresponding evidence will also lead us in a new direction, towards a different interpretation of Dedekind.

In "Was sind und was sollen die Zahlen?" itself a first passage that should give us pause with respect to reading Dedekind as a set-theoretic structuralist is this (already quoted more fully above):

> With reference to this freeing the elements [of a simply infinite system] from every other content (abstraction), we are justified in calling numbers a *free creation* of the human mind. (Dedekind 1988b, p. 68, emphasis added)

Two general questions become central now in connection with this passage: First, how should we understand the notion of "abstraction" in it, as well as the related remark about "freeing the elements from every other content"? Second and crucially, how should we understand the notion of a "free creation of the human mind"?

If we are committed to a set-theoretic structuralist interpretation of Dedekind, a ready answer to the first of these two questions is available. Namely, "abstraction" should be understood thus: Given any simple infinity, we just *ignore* everything about it, and about the elements in it, except for the fact that it forms a simple infinity. In doing so we "free its elements from every other content" *insofar as they are objects of our investigation.* Note that, along such lines, "freeing" has more to do with us, with our perspective or our way of investigation, than with the elements or objects themselves. These objects still have all their other properties; it is just that as objects of our investigation we put them aside.

So far, so good. What about the second question, though, concerning Dedekind's notion of "creation"? Can we, along the same lines, say: What gets created are merely certain *new objects of study*, i.e. objects as seen or investigated by us, not objects in themselves? This answer is less convincing than the answer to the first question, I would suggest. In particular, it does not do justice to the notion of "creation" as used in Dedekind's essay; it makes it too weak a notion, or does not take it seriously enough.

As a matter of fact, our quotation above, involving "creation", is not an isolated remark in Dedekind's writings; there are similar ones elsewhere. Already in the preface to "Was sind und was sollen die Zahlen?" the claim that "numbers are free creations of the human mind" occurs prominently (*ibid.*, p. 31). Further on in that preface Dedekind comes back to talking about "the number-domain created in our mind" (p. 32). Beyond that essay and the case of natural numbers, parallel statements occur especially in connection with the real numbers, both in "Stetigkeit und irrationale Zahlen" (Dedekind 1872) and in related scientific correspondence.

First to "Stetigkeit und irrationale Zahlen". Early on in that essay, even before Dedekind's construction of his famous "cuts", he declares:

> Just as *negative* and *fractional rational* numbers are formed by a new *creation*, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define *irrational numbers* by means of rational numbers alone. (Dedekind 1963, p. 10, emphasis added)

In the first half of this passage, Dedekind is referring to the familiar constructions of both the integers and the rational numbers as equivalence classes of pairs.[17] In connection with both he talks about "creation"; and he sets himself the task to do something parallel for the real or "irrational" numbers. A few pages later, after his introduction of the set of cuts on the rational numbers, he comes back to the same theme:

> Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we *create* a new, an *irrational* number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut. (*Ibid.*, p. 15, first emphasis added)

Note again the emphasis on "creation", and the remark about the new number "corresponding" to the cut, as opposed to being identical with it. Finally, the whole section in which this passage occurs (section IV. of the essay) bears the title "Creation of Irrational Numbers".

Is it possible to interpret Dedekind's construction of the real numbers again along the lines of set-theoretic structuralism? Yes it is – up to a point. The corresponding interpretation is this: In "Stetigkeit und irrationale Zahlen", Dedekind starts with the rational numbers, together with some initial clarifications about the notion of line-completeness or continuity, including the observation that the set of rationals does not have that property. Then he constructs the set of all Dedekind cuts on the rationals so as to show that that set, provided with an appropriate order relation, is line-complete. The crucial step for the proposed interpretation is the next: Dedekind is seen as treating *this very set* as the set of real numbers, thus identifying every real number with a Dedekind cut. Finally, the whole process is seen as involving a process of *abstraction* in the sense of ignoring all the additional properties that cuts have when using them as real numbers.

What I have just described is, of course, exactly the way we proceed in contemporary set theory when introducing "the real numbers". However, is this what Dedekind himself does; in particular, does he take the cuts to be *identical* with the real numbers? No, he does not. Instead, after constructing the set of cuts he notes, first, that not every such cut "corresponds" to a rational number; and then, at this exact point, he makes the remark quoted above about "creation". If we take this remark seriously it means that Dedekind introduces *new* objects, different from the cuts, at this point. Finally, he goes on to form the set of all these new objects, calls *them* "the real numbers", defines an order relation, addition, and multiplication on them, and shows that *that* system is a complete ordered field. This, in any case, is the alternative reading to be considered now.

My sense is that merely based on "Stetigkeit und irrationale Zahlen" we are, perhaps, not forced to give up a set-theoretic structuralist interpretation of Dedekind. After all, one can always play down his remarks about "creation" as colorful asides, not to be taken seriously or literally. However, there are several pieces of additional evidence that refute such an interpretation conclusively. I will consider some of this evidence in the rest of this section, and introduce more in the next section.

A first piece of additional evidence comes from Dedekind's correspondence with the mathematician Lipschitz, from the period immediately following the composition of "Stetigkeit und irrationale Zahlen" (see (Dedekind 1876a), also (Dedekind 1876b)). In one of his letters to Lipschitz Dedekind responds to a number of Lipschitz's questions about his essay on the real numbers. He writes:

> [I]n my essay I show, without bringing in any foreign notions, that in the realm of the rational numbers a phenomenon can be identified (the cut) that can be used, by a single creation of new, irrational numbers, to complete this realm. (Dedekind 1932, pp. 470–471, my translation)

The phrase "foreign notions" is noteworthy here (and will occupy us more later). Note also, once more, the emphasis on the "creation" of "new numbers". But more importantly, Dedekind goes on as follows in a parenthetical clarification:

> If one *doesn't* want to introduce new numbers I have no objections; the theorem proved by me (§5, IV) says then: the system of all cuts in the realm of rational numbers, which in themselves form an incomplete realm, forms a complete manifold. (*Ibid.*, p. 471, my translation, emphasis added).

It seems, then, that Lipschitz had proposed to work with the system of cuts themselves, exactly along set-theoretic structuralist lines. Dedekind, acknowledging that such a procedure is possible, clarifies that it is *not* his own. Rather, according to his own procedure the real numbers are new, separate objects, different from the cuts. In other words, the notion of creation is to be taken more literally and seriously.

6. THE ADVANTAGES OF CREATION OVER CONSTRUCTION

"Stetigkeit und irrationale Zahlen" (1872) and the quoted letter to Lipschitz (1876) were written several years before the publication of "Was sind und was sollen die Zahlen?" (1888). What I did in the previous section was to use evidence from the former two to raise doubts about a set-theoretic structuralist interpretation of Dedekind in the latter, or in general. One may wonder, however, whether that is legitimate. Perhaps Dedekind changed his mind about "creation" in the intervening years?[18] To establish that this is not the case, I will now turn to further parts of his correspondence, in particular to a letter written to the mathematician Heinrich Weber, his friend and sometimes collaborator (Dedekind 1888a). This letter also provides an explanation of *why* Dedekind prefers creation over mere construction.

What makes Dedekind's letter to Weber particularly relevant is that in it both the natural and the real numbers are discussed. Concerning the former, we first find a response to the following suggestion by Weber: Why don't we construct and then *identify* the natural numbers as "cardinal numbers", i.e., as classes of equinumerous classes (apparently along the lines of Frege and Russell, although no details are mentioned)? Dedekind's answer is this:

> If one wishes to pursue your approach I should advise *not* to take the class itself (the system of mutually similar systems) as the number (Anzahl, cardinal number), but rather something new (corresponding to this class), something the mind creates. (Dedekind 1932, p. 489, my translation, emphasis added)

The proposal by Weber concerning the natural numbers is obviously parallel to that by Lipschitz concerning the real numbers. And again, Dedekind makes clear that he prefers to "create something new", in addition to the equivalence classes. Also again, he does not reject the proposed construction outright. Rather, in his characteristically tolerant and supportive way he urges Weber to work out this idea in more detail; as he puts it a little later: "I would recommend very much that you follow, at some point, this line of thought all the way through" (*ibid.*).

The fact that Dedekind himself sees the cases of the natural and real numbers as parallel becomes clear beyond doubt in the passage that follows:

This is *very much the same question* as the one of which you speak at the end of your letter concerning my theory of the irrational numbers, where you say that the irrational number is nothing else than the cut itself; whereas I prefer to create something new (distinct from the cut), something that corresponds to the cut, and of which I say that it produces the cut. (*Ibid.*, my translation, emphasis added)

Both in the case of the natural numbers and the real numbers Dedekind thus explicitly rejects what he is supposed to endorse according to a set-theoretic structuralist interpretation.[19]

But why is it that Dedekind does not want to identify the real numbers with Lipschitz's or his own cuts; likewise, why is it that he does no want to identify the natural numbers with Weber's equivalence classes? The answer to that question is given in the same letter to Weber, in another cryptic, but very interesting remark about the "creation" of numbers:

We have the right to attribute to ourselves such creative powers. In addition, it is much more appropriate to proceed in this way because of the uniformity of numbers. After all, the rational numbers, too, produce cuts, but I will surely not identify the rational number with the cut produced by it. And even after the introduction of irrational numbers one will often talk about cut-phenomena in terms of such expressions, will assign to them such attributes, as *sound strange when applied to the corresponding numbers*. Something entirely analogous also holds with respect to the definition of the cardinal number (Anzahl) as a class. One will say many things about a class (e.g., that it is a system of infinitely many elements, namely all similar ones) that one *would surely attach only very unwillingly to the numbers* (as a weight). Does anyone think about the fact, or does one not quickly and willingly forget, that the number four is a system containing infinitely many elements? (However, that the number 4 is the child of the number 3 and the mother of the number 5 will always and for everyone remain present.) (Dedekind 1932, p. 490, my translation, emphasis added)

Dedekind's argument is this, then: Systems of objects such as Lipschitz's cuts and Weber's equivalence classes simply have *wrong additional properties*, i.e., properties that the real and natural numbers, respectively, do not have; or at least we would want to attribute such properties to these numbers "only very unwillingly".[20]

As formulations like the latter show, Dedekind presents this argument in a somewhat tentative or understated form. It is possible, however, to add more force and depth to it by connecting it with some general, implicit themes in his writings. I have in mind the following: According to Dedekind's general outlook we should aim at keeping the theories of the natural and real numbers as *orderly* and *simple* as possible. Correspondingly, we should aim at keeping the identity and nature of the numbers themselves as *basic* and *pure* as possible. In terms of textual evidence, consider brief Dedekindian remarks such as the following: With respect to the real numbers, geometric and empirical aspects are "foreign notions" (Dedekind 1963, p. 10); with respect to the natural numbers, non-logical and non-arithmetic considerations are "foreign intruders" which "disturb all order"; and overall, Dedekind's goal is, in contrast, to "avoid everything that is disturbing" (Dedekind 1932, pp. 490 and 484, respectively). All of this connects to Dedekind's argument above as follows: Set-theoretic notions, while more closely related to arithmetic ones than geometric or empirical notions, still bring in aspects that are "foreign", "disturbing", and should be avoided.[21]

7. BEYOND PSYCHOLOGISTIC APPEARANCES

In the previous sections I provided reasons for taking Dedekind's remarks about "creation" seriously, both in "Was sind und was sollen die Zahlen?" and elsewhere. Taking them seriously brings with it a danger, though. It is the danger of saddling Dedekind with unattractive, even untenable views, especially crude *psychologistic* views. In fact, interpreters who don't see Dedekind as a set-theoretic or modal structuralist have tended to go in that direction, i.e., have attributed psychologistic ideas to him. We need to consider this issue further then, both to determine whether a psychologistic reading of Dedekind is unavoidable and to clarify what he meant, or could have meant, by "creation".

The psychologism in question involves one or both of the following theses: (i) that the "abstraction" and the "creation" Dedekind talks about are fundamentally psychological processes; (ii) that numbers, as the result of such processes, turn out to be psychological, or mental, objects.[22] Reflecting back on some of the quotations above, it seems hard to avoid attributing especially the second of these claims to Dedekind. Consider also the following additional remark, again from his letter to Weber:

We are of divine species and possess, without doubt, creative powers not just concerning material things (railroads, telegraphs), but especially concerning mental things. (Dedekind 1932, p. 490, my translation)

Clearly numbers are mental things for Dedekind, just as railroads and telegraphs are physical things, aren't they?

What is it that makes such a view unattractive? An initial difficulty, familiar from the critical discussion of such views in (Frege 1884), is this: If numbers are mental things, created by a psychological process in someone's mind, they seem to have an "owner", in the same way that a sensation of pain, say, has an owner. But that means that numbers are always someone's numbers, just as pain is always someone's pain. There are several embarrassing questions that ensue: First, this would imply that there are actually many different systems of numbers: one in the mind of every person who thinks about them. But then, when I assert that "2 + 3 = 5" I am saying something about *my* numbers, while when you assert it, you are saying something about *your* numbers. In other words, statements of number are relative to who makes them, or to whose numbers we are talking about. In itself such relativity may not be absurd or impossible, as a comparison to the relativity in set-theoretic structuralism indicates. There, too, what we mean by "the natural numbers" is relative, not absolute. So perhaps we just have to accept a psychologistic variant of such a view, one where I always pick my *mental* model of the Dedekind-Peano Axioms and you yours?

There are additional questions and problems, however. For example, the suggestion just made only works if we *know* that the different mental models of various people are all isomorphic. But how do we know that? Note, moreover, that such a view requires the existence of infinitely many objects in people's minds, since the system of natural numbers is a simple infinity. But is it really plausible that I, or anyone, can hold so many objects in mind? And even if we grant that, how can we know what is true about *all of them*; perhaps by surveying the whole infinity in our minds? A somewhat different nest of problems is this: Numbers are supposed to have been created in people's minds. But then, *when* exactly were "the natural numbers" created? Was it when the first person thought about them; or when the first person, perhaps Dedekind, thought about them systematically? If the latter, how systematic did the thought have to be? Finally, is it really plausible that there were no natural numbers before, say, Dedekind's definition, as a psychologistic view implies?

These questions are all hard to answer. Indeed, they are strange questions – they look like pseudo-problems, problems we are led to by a misguided view. If we go back to Dedekind's writings, we can see that he brings up some of them himself, at least tangentially. For example, in another letter to Lipschitz (Dedekind 1976b) he writes:

> I did not mean [in 'Stetigkeit und irrationale Zahlen'] that I created, through my definition of irrational numbers, any numbers whatsoever which had not already been grasped before, more or less distinctly, in the mind of every mathematician. (Dedekind 1932, p. 475, my translation)

And in the preface to "Was sind und was sollen die Zahlen?" he remarks:

> But I feel conscious that many a reader will scarcely recognize in the shadowy forms which I bring before them his numbers which all his life have accompanied him as faithful and familiar friends (Dedekind 1963, p. 33).

In both of these passages Dedekind seems to distance himself from an overly literal understanding of the notion of mental creation, like the one underlying crude psychologistic views. Also, in both he doesn't seem to take the matter to constitute a real problem; he treats it rather lightly. This makes it unlikely that he really held the problematic view, unless one is willing to attribute a very superficial mind to him.

There are further reasons to doubt that Dedekind held crude psychologistic views of the kind discussed above. Suppose, for the sake of the argument, that for him numbers are mental entities, i.e., part of various people's mental content. What, then, about the additional properties with which this endows them? For example, a number is now always "owned" by some person. Thus we can attribute a "mental location" to it: it is in my mind, or in your mind, etc. Similarly, we can attribute a location in time to it: the time when it came into existence in the relevant person's mind; also, presumably, the time when it will go out of existence again. But now we can turn one of Dedekind's own arguments against such a view. Namely, are properties having to do with, say, temporal and other locations not again the wrong kind of properties to attribute to numbers; do they not again bring in "foreign notions"? Moreover, is such a view not in clear conflict with programmatic statements by Dedekind such as the following:

> [I] consider the number concept to be entirely independent of the notions or intuitions of space and time. (Dedekind 1963, p. 31)

It is possible, again, that Dedekind was simply inconsistent on this issue, i.e., did not realize the internal conflict between two of his views. More plausibly, this is evidence that he did not hold one of them.

Perhaps what we need is a more sophisticated variant of psychologism for interpreting Dedekind? To ascribe one such variant to him has, indeed, been attempted in the secondary literature as well. The suggestion is to interpret Dedekind's remarks about "the mind" along Kantian transcendental idealist lines, in the sense of making them part of an investigation of the transcendental preconditions of all thinking.[23] Then again, it is not immediately clear whether or how such a suggestion helps with respect to the notion of "free creation".[24]

Such an interpretation of Dedekind may seems attractive in other respects, too. Consider especially the following passage, again from the preface to "Was sind und was sollen die Zahlen?":

[Numbers] serve as a means of apprehending more easily and more sharply the difference of things. [...] If we scrutinize what is done in counting an aggregate of things, we are led to consider the ability of the mind to relate to things, to let a thing correspond to a thing, or to represent a thing by another thing, an ability *without which no thinking is possible*. Upon this unique and therefore *absolutely indispensable foundation* [...] must, in my judgment, the whole science of number be established. (Dedekind 1963, pp. 32–33, emphasis added)

The italicized phrases certainly have a Kantian ring to them, don't they?

Does Dedekind, in passages such as these, talk about abilities of "the mind" in a specifically Kantian transcendental sense? It is hard to be sure, since he says so little about it. He certainly does not explicitly affirm that his view is Kantian, in this or other respects. In addition, he explicitly rejects any more particular Kantian claim about the role of intuition in arithmetic, emphasizing instead the connection between arithmetic and logic:

In speaking about arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result of the laws of thought. (*Ibid.*, p. 31)

Of course, this passage leaves us with the question of how to understand Dedekind's position on logic, in particular on the nature and status of basic "laws of thought". At the same time, what it suggests, at least indirectly, is this: Perhaps it is possible to understand Dedekind's notions of abstraction and free creation in a *logical*, rather than a psychological, manner.[25]

8. INFLUENCES: DIRICHLET, RIEMANN, AND CANTOR

If we reject a crude psychologistic reading of Dedekind and remain skeptical about a Kantian reading, at least in any narrow sense involving intuition, what alternatives remain? In particular, what else could possibly be meant by "free creation" and by Dedekind's other psychologistic sounding phrases? In response, I want to bring into play three new considerations: one philosophical, one mathematical, and one logical. The third of them will be dealt with in the next section; first to the philosophical and mathematical considerations.

The nineteenth century was certainly a period during which both psychologistic and Kantian transcendental idealist ideas were discussed widely, especially in Germany. But it was also a time during which materialist and physicalist claims were put forth forcefully. Consequently many of Dedekind's contemporaries were interested in understanding the difference (or, for materialist, the lack of difference) between the mental and the physical. Against this background, Dedekind's remarks about "abstraction" and "creation" may be seen as making clear the following about his own position: Numbers are *not* to be found on the side of the physical; they are not, e.g., aggregates of physical things located in space and time. Understood as such, Dedekind's remarks have a negative thrust.

Related to this metaphysical issue about the nature of numbers is an epistemological one. Usually physicalist and materialist positions are accompanied by empiricist epistemologies. Now, for a strict empiricist (e.g., J. S. Mill) all our knowledge and all our concepts are based on empirical observation. Seen against that background, Dedekind's emphasis on "free creation" may again be understood to have a negative thrust, namely to be directed against such empiricist views about mathematics, in particular about arithmetic. According to Dedekind, our knowledge in arithmetic is exclusively conceptual and logical, i.e., based on pure thought, not on empirical observation. Note also that, if this is correct, the "abstraction" Dedekind invokes with respect to mathematics is rather different from what empiricists, from Mill all the way back to Aristotle, mean by that term: a process that keeps even arithmetic knowledge crucially tied to physical objects and to empirical knowledge of them. Dedekindian abstraction must be understood quite differently.

The two-fold philosophical suggestion so far – to understand "free creation" primarily in a negative sense, as the rejection of both materialist and empiricist views about mathematics – is not meant to exhaust what Dedekind means by that phrase. There is also an important positive side to it. Turning to this positive side now, it will be illuminating to place Dedekind's discussion in the context of certain debates in nineteenth century mathematics, in particular debates about *mathematical methodology*. These debates involve several well-known mathematicians with whom Dedekind was directly associated: Dirichlet, Riemann, and Cantor, on the one hand, and Kronecker, on the other.

As has been established in the recent literature, Dedekind was strongly influenced by Dirichlet's, Riemann's, and Cantor's works in his general conception of and approach to mathematics.[26] As far as Dirichlet is concerned, the strong and direct ties to Dedekind are clear, at least in general: He studied under Dirichlet (and Riemann) at the University of Göttingen. Later he edited and published Dirichlet's influential *Vorlesungen über Zahlentheorie*. In addition, Dedekind's Nachlaß (stored at the University of Göttingen, Germany) contains evidence that connect Dirichlet directly to our specific concerns.

The main piece of evidence I want to draw attention to in this connection is the following: In Dedekind's Nachlaß one can find several *drafts* of "Was sind und was sollen die Zahlen?". One of them, the most mature one called "Dritter Entwurf" (Dedekind 1887), starts thus:

Numbers are independent of space and time; they originate directly in the laws of human thinking. (Dirichlet, *Vorlesungen über Zahlentheorie*, dritte Auflage, 1879, §163. Comment on p. 470.) (Dedekind 1887, p. 1, my translation)[27]

Clearly this passage is a direct predecessor of the passage from the published version of Dedekind's essay quoted earlier. And in this draft Dedekind makes clear, in his parenthetical reference, what the source of his "logical" conception of the natural number is: Dirichlet's works.

In the case of Riemann, the main influence on Dedekind for our purposes is this: In several of Riemann's works, especially those in what is now called differential geometry, he had made clear how fruitful it is to explore new "conceptual possibilities", e.g., various kinds of non-Euclidean geometries.[28] Crucially for us, at this point in the nineteenth century such geometries were not connected with any direct applications in the empirical sciences (unlike later, with Einstein's work). Their study was not, or at least not directly, driven by empirical concerns, but rather by conceptual and inner-mathematical concerns. It is the latter that characterizes Dedekind's work as well. Moreover, Cantor's main influence on Dedekind can be seen to be along the same lines: Cantor's new set theory, including his theory of transfinite numbers, also went far beyond any direct empirical considerations. They, too, concerned the exploration of "conceptual possibilities" within pure mathematics.

Dirichlet's, Riemann's, and especially Cantor's approaches to mathematics were not uncontroversial, both at the time and later. In fact, they were considered to be radical by many, thus in need of both clarification and defense. In terms of clarification, Cantor writes in his "Grundlagen einer allgemeinen Mannigfaltigkeitslehre" (Cantor 1883):

Mathematics is in its development entirely *free* and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. (Cantor 1883, section 8, paragraph 4, emphasis added)

And in terms of defense, Cantor offers the following:

It is not necessary, I believe, to fear, as many do, that these principles present any danger to science. For in the first place the designated conditions, under which alone the *freedom* to form numbers can be practised, are of such a kind as to allow only the narrowest scope for discretion. Moreover, every mathematical concept carries within itself the necessary corrective: if it is fruitless or unsuited to its purposes, then that appears very soon through its uselessness, and it will be abandoned for lack of success. But every superfluous constraint on the urge to mathematical investigation seems to me to bring with it a much greater danger, all the more serious because in fact absolutely no justification for such constraints can be advanced from the essence of the science – for the *essence* of *mathematics* lies precisely in its *freedom*. (*Ibid.*, paragraph 5, first emphasis added)

Cantor's strong, colorful rhetoric in such passages was directed primarily against mathematicians such as Leopold Kronecker who rejected his theories of transfinite numbers and sets from within mathematics. However, it can also be seen as directed against materialist and empiricists philosophers of mathematics, or against anyone who wants to restrict mathematics to what is directly applicable in the empirical sciences. There should be no such "constraints" according to Cantor; mathematics should be "free".

Dedekind was both in correspondence and friendly with Cantor. Furthermore, not only was he an early supporter of Cantorian set theory, he even contributed various ideas and results to it himself; his definition of infinity in "Was sind und was sollen die Zahlen?" is an example. Of course, Dedekind's essay itself mainly concerns arithmetic, a field Kronecker accepted and studied himself. Thus with respect to the choice of subject matter there is no conflict between Dedekind and Kronecker, unlike in the case of Cantor and Kronecker. Nevertheless, with the very way in which he developed this subject matter, or provided conceptual foundations for it, Dedekind took sides – against Kronecker and for Cantor. Indeed, Dedekind's essay can be seen as a sophisticated attempt to apply Cantor's, Riemann's, and Dirichlet's conceptual or logical approach within the area dearest to Kronecker's heart: arithmetic.

If this is correct, new light is shed on Dedekind's remarks about "free creation". They now reveal themselves also as the endorsement of a certain mathematical methodology: an approach to mathematics that is opposed to both empiricist and materialist and to Kroneckerian strictures, and that, more positively, encourages the "free" exploration of conceptual possibilities as exemplified in the works of Riemann and Cantor.

Actually, Dedekind himself makes his opposition to Kronecker explicit in "Was sind und was sollen die Zahlen?". This occurs in connection with his discussion of the notion of set, or "system". He introduces that notion as follows:

[A system S] is completely determined when with respect to every thing it is determined whether it is an element of S or not. (Dedekind 1963, p. 45)

Obviously, this remark commits Dedekind to an extensional conception of sets. But in a footnote he goes further:

In what manner this determination is brought about, and whether we know a way of deciding upon it, is a matter of indifference for all that follows; the general laws to be developed in no way depend upon it; they hold under all circumstances. I mention this expressly because Kronecker not long ago [. . .] has endeavored to impose certain limitations upon the free formation of concepts in mathematics which I do not believe to be justified. (*Ibid.*, footnote)

Note, once more, Dedekind's emphasis on "freedom", or on the "free formation of concepts". This emphasis occurs directly in connection with an endorsement of unrestricted *classical* mathematics, as opposed to its more constrained constructivist or finitist alternative as advocated by Kronecker.

9. THE LOGICAL DETERMINATENESS OF NUMBERS

There are still two aspects of Dedekind's views that need to be made explicit before I can come back to the issue of structuralism. The first of them has to do with how he thinks about the notion of object, especially in the context of mathematics; the second has to do with the question of consistency. Both of them concern, explicitly or implicitly, *logical* aspects of Dedekind's approach. I will deal with the first in this section and with the second in the next. After that, we will be in a position to provide a logical reading of Dedekind's notion of abstraction.

We observed earlier that for Dedekind the notions of set and function are basic notions for mathematics, indeed for thinking in general. Now, the notion of object is at least as basic for him, since both individual natural numbers and the set of all natural numbers are objects, or "things", for him. Dedekind elucidates his notion of thing at the very beginning of "Was sind und was sollen die Zahlen?":

> 1. In what follows, I understand by *thing* every object of our thought. In order to be able easily to speak of things, we designate them by symbols, e.g., by letters [...]. A thing is completely determined by all that can be affirmed or thought about it. (Dedekind 1963, p. 44, original emphasis)

In this passage, the last sentence will be especially important for us: "A thing is completely determined by all that can be affirmed or thought about it." To be able to appeal to it briefly later, let me give the principle expressed in it a name; let me call it Dedekind's *principle of determinateness* for objects.

Dedekind illustrates this principle immediately by means of the case of "systems", i.e., sets:

> 2. It frequently happens that different things, $a, b, c, \ldots$ can be considered from a common point of view, can be associated in the mind, and we say that they form a *system* S. We call the things $a, b, c, \ldots$ *elements* of the system [...]. Such a system S (an aggregate, a manifold, a totality) as an object of thought is likewise a thing (1). It is completely determined when with respect to every other thing it is determined whether it is an element of S or not. (*Ibid.*, p. 45, original emphasis.)

The last sentence here contains again Dedekind's affirmation of an extensional view about sets.

But there is more to this second passage. Note, first, that Dedekind talks about things being associated "in the mind", which again conjures up psychologistic views, at least superficially. At the same time, the end of the last sentence is exactly where Dedekind adds the footnote directed against Kronecker. In it he emphasizes, as we saw, that it does not matter "whether we know a way of deciding" certain mathematical truths; since "the general laws to be developed in no way depend upon it; they hold under all circumstances" (*ibid.*). What really matters for Dedekind is which "mathematical laws" hold, and *that* depends on the basic definitions and principles involved, as well on what follows from them. What is affirmed by Dedekind, once more, is a *classical* point of view, as opposed to a constructivist or finitist one. In addition, note that the basic principles involved seem to be granted a more *objective* status by him than crude forms of psychologism would suggest.

Back to the notion of object or thing, though. For Dedekind both numbers and sets are things, even "completely determinate" things. Now, what exactly is the sense in which numbers, in particular the natural numbers, are completely determinate according to him? Applying Dedekind's general principle of determinateness, we are led to the following: It has to be completely determined "what can be affirmed" concerning them. But how is that accomplished? To answer that question we have to consider Dedekind's whole procedure in "Was sind und was sollen die Zahlen?", since this is really what the essay is all about, as the first part of its title ("What are numbers ... ?") already indicates.

The way in which Dedekind's procedure "determines completely" what numbers are consists of three basic parts: (i) He specifies a language in which we can say things about the natural numbers. More specifically, he tells us how to use the two symbols '1' (for the first natural number) and '$\phi(x)$' (for the successor function) to define all the other arithmetic notions needed. (ii) He formulates the basic definitions and principles from which all the truths about numbers – thus all that can be "affirmed" about them – are supposed to follow. (iii) He proves that the system so specified is categorical. How, or in what sense, does this procedure accomplish what Dedekind wants? As follows: By (iii), for any sentence φ in the language of arithmetic, as specified in (i), either φ or $\neg\varphi$ follows (semantically) from the basic definitions and principles, as specified in (ii); *tertium non datur*. In other words, Dedekind's theory for the natural numbers is (semantically) *complete*, as follows directly from its categoricity.[29]

This treatment of "determinateness" for the natural numbers is subtle and ingenious, both in a mathematical and a philosophical sense. The whole procedure depends, of course, on Dedekind's theory being *consist-*

ent; since otherwise for every arithmetic sentence both it and its negation would follow. In "Was sind und was sollen die Zahlen?" Dedekind is not as explicit about consistency as he is about categoricity and completeness. Still, there is one part of the essay that can be seen to be directed at this issue, if only implicitly. Dedekind's "proof" of the existence of an infinite system, thus also of a simply infinite system, can be taken to play the role of a semantic (as opposed to syntactic) consistency proof.

At it turns out, there is evidence that this is exactly the role Dedekind himself saw for this proof, even if he doesn't say so explicitly in "Was sind und was sollen die Zahlen?" This evidence can be found in a well-known letter to the teacher Keferstein (Dedekind 1890). In that letter, Dedekind explains the basic procedure in his essay as follows:

> After the essential nature of the simple infinite system, whose abstract type is the number sequence *N*, had been recognized in my analysis (articles 71 and 73), the question arose: does such a system exist at all in the realm of ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. *Hence the need for such proofs* (articles 66 and 72 of my essay). (Dedekind 1890, p. 101, emphasis added)

Dedekind's construction of an infinite system in his essay is meant, we are told, as a "logical proof" of consistency. As such, it complements his presumably equally "logical proof" of categoricity.

Some *caveats* are immediately in order here: First, Russell's antinomy has of course undermined the basis for Dedekind's proof of consistency, as mentioned above and as he acknowledged himself later. Second, from Gödel's work we know that there are even deeper problems concerning the provability of consistency for arithmetic. What that means, in our context, is the following: What Dedekind *actually* achieved is less than what he *thought* he had achieved. Furthemore, there are *profound reasons* for his failure. Nevertheless, it would be wrong to say that Dedekind's project was a complete failure. He did achieve a considerable and impressive part of what he wanted.

As this may be, we have arrived at a better understanding of Dedekind's position, especially of his conception of number. This conception consists of three main ingredients: (i) a precise specification of the language in which to make arithmetic assertions; (ii) an explicit list of definitions and basic principles for arithmetic formulated in this language; (iii) (attempted) proofs that these definitions and principles form a theory that is both consistent and categorical. Crucially for us, had Dedekind been able to secure all three of these ingredients, this would, in his own view, have constituted the following: an explicit and precise explanation of the sense

in which the natural numbers form a *system of completely determinate objects*; moreover, an explanation *in purely logical terms*.

10. A COMPARISON: DEDEKIND AND HILBERT

The notion of "determinateness" as discussed in the previous section – with emphasis on its logical character in the case of numbers – is intimately connected with Dedekind's notions of "abstraction" and "free creation". A comparison of Dedekind's views to similar ones expressed David Hilbert not long thereafter will make this connection evident.

The text by Hilbert on which I want to focus is his short article "Über den Zahlbegriff" (Hilbert 1900).[30] In it he formulates a system of axioms for the real numbers. These axioms are basically those for a complete ordered field. Dedekind had earlier, in "Stetigkeit und irrationale Zahlen", presented a very similar characterization of the real numbers, but had not been as explicit as Hilbert about the various axioms involved.[31] Crucially for present purposes, Hilbert, too, brings up both *consistency* and *completeness* as desiderata.

First to the issue of consistency. Unlike in his later works, in "Über den Zahlbegriff" Hilbert is still rather vague about the precise form a consistency proof for his axioms is supposed to take. He just declares, rather cryptically: "In order to prove the consistency of the specified axioms, all one needs is a suitable modification of known methods of inference" (Hilbert 1900, p. 184, my translation). It is not clear whether he has a semantic or a syntactic proof in mind here, although the reference to "known methods of inference" suggests the former. In any case, Hilbert is more explicit about the *significance* of such a proof:

> This proof [of consistency] I also regard as the proof for the *existence* of the set of all real numbers, or – to use G. Cantor's formulation – the proof that the system of real numbers is a consistent (complete) set. (*Ibid.*, p. 184, my translation, emphasis added)

Thus, existence in mathematics is tied directly to consistency by Hilbert. Indeed, nothing further is required for it according to him.[32]

With remarks such as these, Hilbert puts himself directly in the tradition of Dirichlet, Riemann, Cantor, and Dedekind, as discussed above. Actually, his whole approach in the acticle under consideration does so: First, he presents an abstract, conceptual, or logical approach to the real number, in his case one that is explicitly axiomatic. Second, he states that all that is needed in addition is to establish the consistency and the completeness of the axioms (as well as the fact that, just as in Dedekind's treatment of the natural numbers, all the usual truths about the real numbers follow).[33]

Finally, he draws the following conclusion with respect to the *nature* of the real numbers:

So we must think of the set of real numbers not as, say, the totality of all possible laws for laying out the elements of a fundamental sequence, but rather – as just explained – as *the system of things whose relationships have been determined by the finite and complete system of axioms I-IV above*, and about which an assertion is only valid if it can be derived, in a finite number of logical steps, from those axioms. (*Ibid.*, p. 184, my translation, emphasis added)

Note that for Hilbert the real numbers are a particular system of objects, namely those objects whose identity and nature have been "determined completely" by the axioms and what follows from them. This way of thinking about the real numbers parallels exactly Dedekind's conception of the natural numbers. And as Hilbert was quite familiar with Dedekind's work, this parallel is surely no accident.

What passages such as those above show, more particularly, is that Hilbert adheres explicitly to what I have called Dedekind's *principle of determinateness*. Note that, like Dedekind earlier for the natural numbers, Hilbert provides the following for the reals: (i) a particular language in which to formulate statements about them; (ii) a set of axioms from which to derive truths about them; and (iii) observations about the completeness, indeed categoricity, of this set of axioms. Once more, these three things together are supposed to determine a particular system of objects, "the real numbers". How or why? Because it is then completely determined what can be asserted about the elements of it, i.e., which truths hold concerning them; in Hilbert's words again: "[A]n assertion is only valid [about them] it if can be derived, in a finite number of logical steps, from these axioms".

Having noticed these similarities, there appears to remain an important difference between Hilbert and Dedekind. Namely, Hilbert does not talk about "abstraction" or "free creation"; he simply presents his system of axioms and tells us to focus on what follows from them. This may be the reason why Hilbert, unlike Dedekind, has seldom been interpreted as a structuralist.[34] He is usually seen as a formalist, an impression reinforced by his own explicit emphasis on the formal aspects of mathematics in his later writings. But how deep does this difference go? To answer that question we need to extend the comparison between Dedekind and Hilbert a bit further.

Remember that Dedekind first constructs an infinite set, as a subset of the "totality of his objects of thoughts", and then a simply infinite set as a subset of that. Given this latter system, he invokes a process of "abstraction" and "free creation" to obtain "the natural numbers". Hilbert, on the other hand, does not rely on a corresponding initial construction in the case of the real numbers. Or at least he does not do so explicitly – perhaps his brief aside on "known methods of inference" to establish the consistency of his set of axioms is a reference to such constructions, including Dedekind's construction of Dedekind cuts out of rational numbers? Whether we interpret Hilbert's cryptic aside along those lines or not, there is a way to reconcile their two approaches that is illuminating and relevant.

Both Hilbert and Dedekind are centrally concerned about "determining completely" what holds and what doesn't hold about their respective system of objects. Both present particular, restricted languages, then categorical systems of axioms in those languages to do so. Now, given such a categorical axiom system, there are two ways to think about the determination in question: (a) We construct a model of the axioms and say that the truths are exactly those truths in the language that hold in *that* system, knowing full well that the same truths will also hold in any other model. Or (b), we say that the truths are those that follow semantically from the axioms, i.e., those sentences true in *all* models. The first method is Dedekind's, the second Hilbert's (assuming that Hilbert means "derivability" from the axioms in a semantic sense, or at least as compatible with such a sense, as seems reasonable). How different are they in the end? Well, if we grant the existence of a model the two methods are completely equivalent in this crucial respect: they determine exactly the same truths.

11. DEDEKIND'S LOGICAL STRUCTURALISM

Having set aside a crude psychologistic reading, earlier we interpreted Dedekind's remarks about "abstraction" and "free creation" as, at least in part, an expression of two related points: the rejection of materialist and empiricist views about mathematics, views that tie mathematics too closely to its applications; the endorsement of the mathematical methodology Dedekind encountered in Dirichlet's, Riemann's, and Cantor's works, including the rejection of Kroneckerian strictures. These two points give the notion of "freedom" in mathematics both a negative and a positive content. However, more needed to be said about the notions of "creation" and "abstraction", especially on the positive side. Now we are in a position to do so. More particularly, I now want to present an interpretation of "abstraction" as a *logical* method or process.[35]

In which sense is what Dedekind does "logical", especially when he appeals to "abstraction"? Let us go back to the crucial step as discussed above. After having constructed a simple infinity, Dedekind tells us to "abstract away" from all the non-arithmetic properties of the objects in it. All that matters, instead, are the arithmetic truths that hold in it, i.e., those truths expressible in the particular language specified by him. To justify this move Dedekind shows, shortly thereafter, that all simple infinities are isomorphic, so that exactly the same arithmetic truths hold in all of them. Now, our comparison with Hilbert has revealed that from a logical point of view the following three aspects are crucial: It is specified what kinds of sentences can be formed; it is determined which of them count as true; and it is shown that this has been done in such a way that the law of bivalence holds: a sentence either is true (follows semantically from the axioms) or its negation is true.

To make what is crucial here really clear, note the following: First, the way the truth or falsity of arithmetic sentences is "determined" in this process is not psychological, but logical, in the sense that what counts is what follows logically (in the semantic sense) from our basic principles. Second, the way we "abstract away" from all non-arithmetic properties of objects in the original simple infinity is logical, too, insofar as what we do is to restrict ourselves to what can be expressed in terms of certain fundamental notions. Third, the sense in which a simple infinity is "created" in the process is this: It is *identified* as a new system of mathematical objects, one that is neither located in the physical, spatio-temporal world, nor coincides with any of the previously constructed set-theoretic simple infinities. To put the last point slightly differently, what has been done is to *determine uniquely* a certain "conceptual possibility", namely a particular simple infinity. Which one again, i.e., what is the system of natural numbers now? It is that simple infinity whose objects only have arithmetic properties, not any of the additional, "foreign" properties objects in other simple infinities have.[36]

Such an interpretation attributes to Dedekind the view that the natural numbers are a system of *sui generis* objects. They are *sui generis* in the sense of being different both from ordinary physical objects and from objects in other simple infinities in mathematics, such as those constructed in pure set theory. Moreover, along these lines the nature and identity of all the natural numbers is determined *together*. That is to say, it makes no sense to talk about one natural number alone to even have an identity of a nature (in isolation, completely independently from the others). The reason is that the language and the truths for the natural numbers are determined *together*. What Dedekind identifies, in other words, is the "structure" of the natural numbers as a whole. This is an important part of the sense in which such a position amounts to a kind of "structuralism". Finally, for reasons that should be clear by now I propose to call Dedekind's particular position *logical structuralism*.

Evidently this kind of structuralism is different both from set-theoretic and modal structuralism. On the other hand, it does show some similarities to *ante rem* structuralism, doesn't it? Indeed, *ante rem* structuralism is the position that comes closest to Dedekind's in the current literature. However, there also remain some important differences between these two positions. I now want to discuss the two main differences briefly. The first concerns the claim in *ante rem* structuralism that the system of natural numbers, say, is a "universal". The second concerns the claim that this universal exists in some "platonic" sense. My general conclusion will be that Dedekind's position, if looked at carefully, is different from *all* the main versions of structuralism in the current literature.

According to Stewart Shapiro's *ante rem* structuralism a structure such as that formed by the natural numbers is a "one-over-many". As he also puts it: "Structure is to structured as pattern is to patterned, as universal is to subsumed particular, as type is to token" (Shapiro 1997, p. 84). While I do not find it easy to understand the real force of statements such as these, it is implied, presumably, that there is a significant difference between particulars and universals. Both the whole structure of the natural numbers and the individual numbers are, then, decidedly *not* particulars according to Shapiro. This is further reflected in his terminology: he talks about individual numbers as mere "places" or "positions" (*ibid.*, p. 83 etc.); they are mere "shadows" of particular objects, so to speak.

As in the case of set-theoretic structuralism, there are some passages in Dedekind's writings that, at least on a first glance, appear to show that he held exactly such a view. In this case it is a passage in Dedekind's letter to Keferstein, as already quoted above, that stands out. In that passage, Dedekind talks about the system of natural numbers as introduced by him as the "abstract type" of all simply infinite systems (see again (Dedekind 1890), p. 101). Thus he seems to invoke some kind of type-token distinction. Two different observations put a reading of Dedekind along Shapiro's lines immediately in doubt, however. First, Dedekind has no qualms to talk about individual natural numbers as "objects" or "things", as we already saw. Likewise, the natural numbers as a whole are called a "system", which in turn is a *bona fide* "thing" for him as well (as long as it is "completely determined"). Second, except for the one letter in which he uses the term "abstract type" once, Dedekind never, to the best of my knowledge, talks about the relation of a set-theoretic simple infinity to the natural numbers as that of a particular to a universal under which it falls. Rather, their relation is presented as that of two isomorphic systems. This suggests that the natural numbers are on the same level, or in the same category, as other systems and their elements; they are all "things".

Perhaps what these two observations suggest is that the difference between Dedekind and an *ante rem* structuralist like Shapiro lies not so much in their respective notions of *structure*; it seems that these are pretty similar. Rather, the difference between them lies more in their respective notions of *object*. As we saw, Dedekind works with a logical notion of object, as tied to his principle of determinateness. Moreover, this is a notion that is in a sense tailor-made for the mathematical case. Shapiro, on the other hand, seems to base his discussion on a notion of object for which ordinary physical objects, or at least "non-structural" objects, are the paradigmatic example. Consequently, numbers are not objects for him but something less, something more shadowy.

At the same time, Dedekind and Shapiro agree that both individual natural numbers and the structure of all natural numbers are "independent" from all physical and other non-structural objects. This brings us to the second difference I want to point out, having to do with the *platonist* side of Shapiro's views. He writes: "Structures *exist* whether they are exemplified in a non-structural realm or not" (Shapiro 1997, p. 89, my emphasis). The emphasis on "existence" here corresponds to Shapiro's use of the qualifier "*ante rem*", as opposed to "*in re*", for his brand of structuralism. He also likens his position explicitly to that of Plato, thus invoking the stereotypical "platonist" interpretation of Plato's theory of forms.[37]

With respect to this aspect, too, I do not find it easy to be sure about what is really at issue. According to our earlier discussion, what is crucial for Dedekind in connection with the natural numbers is the consistency and categoricity, thus completeness, of his axioms, or rather of his notion of a simple infinity. And as our comparison to Hilbert has brought out, this leads to what has traditionally been considered the core of a "formalist" view: the claim that existence in mathematics amounts to nothing more than consistency, at least in the case of a complete theory. Now, if Shapiro just means *that* by "independent existence" and by "platonism", then his and Dedekind's views are quite close. However, Shapiro's somewhat heavy-handed use of the term "platonism" (similarly, that of "realism") calls such a "formalist" interpretation of him into question.

Going back to our earlier discussion, if we understand Dedekind's remarks about "mental creation" in an overly literal sense, for example in a crude psychologistic sense, there is obviously a big difference between Dedekind and Shapiro. The natural numbers will then be mental objects for Dedekind, while they are certainly not mental objects for Shapiro. But if, instead, we understand "creation" and "abstraction" in the logical sense defended in this essay, it follows that Dedekind's notion of object is *fairly minimal*. An object, or thing, for him is then just something determinate enough to be *the object of systematic investigation*. Nothing further, especially nothing more metaphysical, is involved, neither in a psychologistic nor in a substantive "platonist" sense.

12. ANTICIPATING TWO OBJECTIONS

According to my reading, for Dedekind the natural numbers are *sui generis* objects, wholly determined in their identity by what follows from the Dedekind-Peano Axioms. This does not mean that we cannot study other models of these axioms as well, i.e., other simple infinities; Dedekind himself, while not using the terminology of "axioms" and "models", does so. What it means, instead, is this: Besides all the set-theoretic models and besides all physical models (if there are any) there is one further, distinguished model. And it is this distinguished model that deserves to be called "the natural numbers". That is to say, we may regard it, or the objects in it, as what we talk about when we *use* arithmetic language in doing arithmetic, as opposed to when we treat the language of arithmetic as an uninterpreted formal language in contemporary set theory and model theory.

Having argued for such an interpretation, it remains to be asked whether the position we have arrived at – logical structuralism – is *tenable*, i.e., defensible in itself. While I do not want to claim that the position is without problems, I will try to show that it is defensible, indeed attractive, in a number of respects. In this section I want to bring up two objections that have been raised against structuralism in general and that may seem particularly acute in the case of logical structuralism. These two objections concern, respectively: the existence of numbers; their identity. I think that neither constitutes a major problem for Dedekind's position as developed here. In the subsequent section I will bring up a third objection, concerning the determinateness of numbers, that I consider to be more serious.

First once more to the notion of *existence*. According to my interpretation a Dedekindian logical structuralist works with notions of object and existence that are fairly minimal. In Dedekind's words again: a "thing" is "every object of thought"; and to be an object of thought only requires to be "completely determined" with respect to "all that can be affirmed or thought about it" (Dedekind 1963, p. 44). The latter, in turn, can be ensured by working with a theory that is both consistent and categorical. Our first objection consists now in questioning whether this is enough to ensure "real existence" for numbers, i.e., whether the notions of object and existence involved here are "thick" enough.

A logical structuralist can respond to this objection in a number of related ways: First, it is not clear what "real existence" or a "thick" notion of existence amount to, as presupposed in the objection. If the paradigm is existence in the physical world, so that physical objects such as the Moon or the Eiffel Tower are paradigmatic objects, then numbers and other mathematical objects are perhaps deficient. However, to just assume such notions of object and existence seems to simply beg the question against mathematical objects. What is more, it seems to introduce notions that are completely irrelevant for mathematical practice. Second, if we do consider what is relevant for mathematical practice (at least classical mathematical practice, the kind Dedekind was concerned with) we are led exactly to the issues considered by him. In particular, mathematics is fundamentally concerned with inference. But what, besides inconsistency and incompleteness, could really threaten such inference? Third, suppose, for the sake of the argument, that we grant the cogency and significance of some thicker notions of object and existence. Then it should still be clear what logical structuralism amounts to, i.e., which notions of existence and object it uses. And it is not obvious that these notions are internally incoherent, is it?

Perhaps the most convincing response to our first objection is more positive, though. It is to point out the following: What matters with respect to mathematical objects such as the natural numbers is simply how we *reason* about them. Now, Dedekind's procedure determines exactly what is relevant for such reasoning: it specifies the language and the basic principles, or axioms, from which to start; and it shows, in a systematic way, how one can derive further concepts and truths from them. Indeed, Dedekind's approach allows to reconstruct all that is crucial about numbers from a mathematical point of view. It is in this sense that he answers the questions of what numbers are ("Was sind die Zahlen?") and what they can be used for ("Was sollen die Zahlen?"). Moreover, he answers them in a way that doesn't bring in any further, "foreign" elements. Seen from this point of view, it is the notion of "real existence" that is in need of justification in connection with mathematics, not the other way around.

A second objection commonly raised against structuralist views is the following: Do the principles specified really single out *one* system, "the natural numbers"; do they not, instead, merely determine a *class* of such systems?[38] Put slightly differently, can the Dedekind-Peano Axioms really be taken to determine the identity of a *particular* system of objects, as opposed to merely determining natural number systems *up to isomorphism*?

This objection, unlike the first one, is connected with current mathematical practice. In fact, most mathematicians today treat the Dedekind-Peano Axioms exactly thus: as determining progressions, or "ω-sequences", up to isomorphism. And if combined with using ZFC as the framework for studying such ω-sequences, this leads naturally to set-theoretic structuralism. To resist this train of thoughts a Dedekindian structuralist will, consequently, have to go beyond standard mathematical practice. He or she will have to bring in additional philosophical considerations.

To put these additional considerations in context, I want to introduce one last piece of evidence from Dedekind's writings. This piece comes again from his Nachlaß, more precisely from the earlier draft (the "third draft") of "Was sind und was sollen die Zahlen?" already appealed to above (Dedekind 1887). It also takes us back to the argument for a set-theoretic structuralist reading of Dedekind based on the notation used in his essay. As it turns out, in that earlier draft Dedekind makes a clarifying remark about his notation that did not make it into the published version, but is quite illuminating.

The part of Dedekind's essay this takes us back to is the one in which he introduces "the natural numbers" by means of "abstraction" (Definition 73 in the published version). In the draft version, we can find the following clarification at this point:

> As this abstraction transforms the originally considered elements n of N into new elements $\mathfrak{n}$, namely the numbers (and N itself into a new, abstract system $\mathfrak{N}$), one is justified in saying that the existence of numbers is due to an act of free creation by the human mind. However, notationally it is more convenient to refer to the numbers as if they were the original elements of the system N, and to simply ignore the transition from N to $\mathfrak{N}$, which itself is a similar mapping. As one can ascertain by means of the theorems concerning recursive definitions [...], nothing essential is changed along these lines, and nothing is left out in an illegitimate way. (Dedekind 1887, p. 5, my translation)[39]

It is not clear why Dedekind left out this helpful clarification in the published version of the essay. Somehow he must have thought that it wasn't needed, or that it would hurt more than it would help. In any case, two things are evident from it: First, the fact that Dedekind doesn't introduce a different notation for the natural numbers – the "new, abstract system" introduced via "free creation" – does not, in itself, provide conclusive evidence for a set-theoretic structuralist reading; it can be explained in terms of "notational convenience", as he himself does in the draft. Second, Dedekind acknowledges, more or less explicitly, that for mathematical purposes one can just use the original system N, rather than moving over to the new system $\mathfrak{N}$.

But why, then, did Dedekind not edit out his remarks about "free creation" as well in his final, published version of his essay, together with this clarification? Why, in other words, didn't he just, clearly and unambiguously, work with the original system N? Well, we have already encountered his reasons. The main one is that the original elements n of N, or more generally the objects in any set-theoretic model of the Dedekind-Peano Axioms, have wrong additional properties. The natural numbers themselves – the elements of the "pure" series of numbers, as he also puts it in his draft (*ibid.*) – do not have such "foreign" properties; or they should not have them in a proper reconstruction and systematization.

At this point a sharpening of our second objection may suggest itself. It can be put in terms of the following two questions: Does the idea of having objects without any additional, non-arithmetic properties make sense at all? And even if we assume it does, why or in which sense does Dedekind's actual procedure single out such privileged objects, as opposed to just determining various models up to isomorphism? Let me start by giving a brief, initial answer to the second question. Spelling this answer out further will then, in the next section, lead to a response to the first as well.

The brief answer to the second question is this: It clearly has to be acknowledged that the Dedekind-Peano Axioms *as standardly used today*, in contemporary set theory and model theory, do not single out one of their models as privileged. However, Dedekind himself does *not* use these axioms – or rather, his definition of a simple infinity – exactly in this contemporary way. For one thing, he does not, strictly speaking, work with a formal, uninterpreted language. Instead, his main concern is to give the language of arithmetic a *definite, particular* interpretation. Moreover, he does so exactly by appealing to a kind of "abstraction" that is absent from contemporary set-theoretic and model theoretic approaches. Finally, this abstraction has the effect that the natural numbers *only* have arithmetic properties – or at least this is what we have said so far, as a first approximation.

13. A FURTHER CLARIFICATION

Looked at in more detail, this last point cannot be quite right, though; it cannot be that the natural numbers have *no* properties besides the intended arithmetic ones. For example, why isn't it the case, even along Dedekind's lines, that the number 2 has the property of being an element of the set $\{1, 2, 3\}$; that the number 5 has the property of being referred to by the numeral '5'; and that the number 9 has the property of being the number of planets in our Solar System? Likewise, why isn't it the case that the whole system of natural numbers has the property of being a favorite object of study for many mathematicians, as well as the main example for many philosophers of mathematics, including me? Surely even a structuralist *à la* Dedekind would't want to deny such truisms, would she?

These are, of course, again somewhat odd questions, especially from a mathematical perspective. But from a philosophical point of view one wants to have answers to them. I would, thus, like to briefly sketch such answers, even if this will take us beyond what Dedekind himself says. My main suggestion in this connection consists in distinguishing *constitutive* from *non-constitutive* properties. Basically, a constitutive property of an object is one that is tied to its very identity, i.e., one without which the object wouldn't be the object it is, perhaps not even the same kind of object. All other properties are considered non-constitutive.

Let us consider some examples. It is a constitutive property of any *physical* object that it has some location in space and time. The particular spatio-temporal location of such an object, on the other hand, is non-constitutive, since the object may be moved around without automatically loosing its identity. Conversely, it is a constitutive property of any *abstract* object that it has no location in space and time. Numbers, sets, etc., as conceived of by Dedekind, are alike in being not physical, but abstract objects – this is partly what is meant by Dedekind's remark that they are "free creations of the human mind", as we saw above. To differentiate among abstract objects, we can go further: It is a constitutive property of any set except for the empty set that it has elements. In this case it is even a constitutive property which particular elements the set has (given the axiom of extensionality for sets). But it is not a constitutive property of a set that it is an element of some other set. Again, it is a constitutive property of any natural number except for 1 that it is the successor of some other natural number (assuming here that we start the series of natural numbers with 1, as Dedekind does). It is also a constitutive property of the number 4 that it is the successor of the number 3, as well as the predecessor of the number 5 – as Dedekind put it, "4 is the child of 3 and the mother of 5" (Dedekind 1932, p. 490). But it is not a constitutive property of the number 9 that it is the number of planets in the Solar System, also not that it is referred to by the numeral '9' by us. Nor is it a constitutive property of the series of natural numbers that it is a favorite topic of mine.

These are just a few examples, meant to provide an initial sense of what is meant by "constitutive" and "non-constitutive". Is it possible to be more precise, i.e., to say explicitly and systematically what the constitutive properties of certain objects are? Yes, we can – at least in the case of mathematical objects. To start with our main example, the natural numbers, we have already noted that they are abstract objects. Thus they have all the properties that follow from being abstract as constitutive properties, including not being located in space and time.[40] Beyond that and in contrast to other abstract objects, the constitutive properties of the natural numbers are precisely those they have as a consequence *solely* of the fact that they satisfy the Dedekind-Peano Axioms. Similarly, the constitutive properties of the real numbers, in addition to what follows from being abstract, are those properties they have as a consequence *solely* of the fact that they satisfy the axioms of a complete ordered field. Admittedly it is difficult to be as precise when we go beyond such mathematical examples. In fact, it may be impossible to say what exactly the constitutive properties of a particular physical object are. This may even be a decisive difference between mathematical and other objects.

What do we gain from introducing this distinction for present purposes, i.e., how does it clarify our earlier discussion? As follows: Dedekind's invocation of "abstraction" can now be recognized as specifying exactly what the constitutive properties of "the natural numbers" are, as opposed to those of other simple infinities. Consider, for example, the set of finite von Neumann ordinals ω. It, like the system of natural numbers, is an abstract object, as are all the sets that are its elements; and it, too, forms a simple infinity. However, the elements of ω have constitutive properties that are non-arithmetic, simply in virtue of being sets (see above). Similarly, the objects of any other simple infinity, including ones containing physical or mental objects, will have non-arithmetic constitutive properties.

We understand better now what distinguishes the natural numbers from the elements of other simple infinities, or what makes them form a *distinguished* system, a system of *sui generis* objects. But what about the question of whether the natural numbers also have other properties or not? This question can now be answered, too: Yes, the number 9 has the property of being the number of planets in the Solar System. It is just not a constitutive property; the number 9 would still be the particular number it is even if there was an additional planet beyond Pluto. Similarly for the other examples above.

One question, or challenge, remains to be addressed, a challenge we can now reformulate slightly to make it more precise. It is this: Does it really make sense to conceive of objects that have *only* mathematical, structural properties as *constitutive* properties? After all, compared to ordinary physical objects these are odd "objects", aren't they?[41]

From Dedekind's perspective one can respond to this challenge in two related ways: by raising a counter-question; and by formulating a counter-challenge. As to the first, *why* does it not make sense to conceive of such objects, i.e., what exactly is supposed to be problematic about them? As already admitted, they are different from physical objects. But they are still "determinate objects of thought" in Dedekind's sense, i.e., things we can reason about systematically, aren't they? Second, does this objection not, blindly and dogmatically, rely on a notion of object that is simply too narrow, and biased towards the example of physical objects? In other words, does the notion of object *presupposed* in the objection not have to be clarified further before rejecting Dedekind's position? More provocatively, while Dedekind's notion of object is subtle and useful, the one underlying the objection is coarse and vague, isn't it?[42]

14. LIMITS OF LOGICAL STRUCTURALISM

The third and final objection to logical structuralism I want to consider, which will also point us towards its limits, involves three related aspects of Dedekind's position that have come up already, more or less explicitly. They have to do with the issues of definitions, completeness, and consistency, respectively.

Above I pointed out that one part of Dedekind's procedure in determining the nature and identity of the natural numbers is to specify the language in which to make statements about them. This now requires further clarification. It is certainly true that Dedekind specifies, explicitly and exactly, what the basic symbols to be used are, namely '1' (for the first natural number) and 'ϕ' (for the successor function). It is also true, as has remained more implicit so far, that he allows for the *definition* of further expressions out of these basic ones.[43] But what kinds of definitions are allowed? More specifically, what is the background logic in which to work?

From a contemporary point of view we may be inclined to think that this background logic has to be first-order logic. However, a quick look at the definitions Dedekind actually gives, as well as at what they are supposed to accomplish, shows that this is too narrow. Consider, e.g., his recursive definitions of addition, multiplication, and exponentiation; also his use of an informal theory of functions and sets in defining other notions, such as that of a finite initial segment of the number sequence. It is more natural, then, to consider *higher-order logic* as the background, say in the form of the simple theory of types. While Dedekind himself does not do so – the simple theory of types had not been developed yet when he wrote – this would conform to his clear intention of formulating a categorical theory for the natural numbers, which is not possible in first-order logic.

Using higher-order logic allows us to be precise about the kind of definitions used in Dedekind's approach, in a way that conforms to his goals. However, it also leads to new questions and problems, especially concerning the notions of *completeness* and, more basically, that of *logical* *consequence*. To begin with, it follows from Gödel's incompleteness results that the usual notion of semantic consequence for higher-order logic does not coincide, indeed cannot be made to coincide, with any notion of consequence in standard syntactic terms. Moreover, while the categoricity of the Dedekind-Peano Axioms implies their semantic completeness (also in higher-order logic), it does not imply their syntactic completeness, in the sense that for any arithmetic sentence either it or its negation is a syntactic consequence.[44] In fact, Gödel's results show that the Dedekind-Peano Axioms are not syntactically complete, even that they cannot be made complete in this sense.

Dedekind himself did not make a distinction between semantic and syntactic consequence. It took various later developments, culminating in Gödel's and some related discoveries, to establish that making it is logically significant. Now, the fact that this distinction is significant does affect Dedekind's notion of *determinateness*, in the following way: While it is true that his procedure determines for every arithmetic sentence that either it or its negation is a semantic consequence of his basic principles, this does not give us a general way of *deciding syntactically* which of these is the case. More pointedly, for any system of formal deduction there will always be a sentence for which neither it nor its negation follows syntactically from his axioms. In short, Dedekind's natural numbers, while determinate in a semantic sense, are *not* determinate in a syntactic sense.

How damaging a problem is this for Dedekind, or for a logical structuralist more generally? It certainly is a decisive blow if we expected mathematical objects to be determinate not just in the semantic, but also in the syntactic sense, i.e., if we were only willing to regard them as "objects" in that case. The incompleteness results of early twentieth century logic, in particular Gödel's results, establish conclusively that one cannot have a notion of mathematical object that is determinate in that sense, at least if we consider objects as logically rich as the natural numbers. This is just a logical fact.

There are two basic reactions to this fact: One can insist on syntactic determinateness as a criterion for objecthood, with the result that one is forced to give up the conception of natural numbers as objects. Such a move is usually accompanied with restricting oneself to first-order logic, the strongest logic for which the notions of semantic and syntactic consequence coincide. Alternatively, one can relax one's notion of objecthood to allow for syntactic indeterminacy, taking only semantic determinateness as required. In that case we can still consider the natural numbers to be mathematical objects; it's just that, unavoidably, their determinateness is limited as indicated. The second of these options would be more in line with Dedekind's approach, I would suggest, since it allows us to preserve as much of his position as possible.

Finally, back to the issue of *consistency*. Clearly, no mathematician today will accept Dedekind's "proof" of the existence of an infinite set as a satisfactory mathematical proof of consistency for arithmetic. This is so not only because of its apparent psychologistic aspects, but also, and more seriously, because of the lessons learned from Russell's antinomy etc. A few additional clarifications about Dedekind's procedure are in order now: First, I want to suggest that it may be possible to reinterpret Dedekind's appeal to "the totality of things which can be objects of [his] thought" in non-psychologistic terms. Two alternative reinterpretations suggest themselves: (a) we can try to interpret the notion of thought involved here in a (Fregean) non-subjective sense, perhaps backed up by an intensional logic that specifies axiomatically which such thoughts exist; (b) we can try to take the notion of possibility implicit in Dedekind's formulation seriously, and then bring in modal logic in an appropriate way. In addition, notice that Dedekind's procedure to use a thought, then a thought of that thought, etc. in his construction is not so different from the specification of, e.g., the finite Zermelo ordinals in set theory, in which we use a set, then the set including that set, etc. to construct an infinite sequence.

As this appeal to set theory indicates, there remains a problem, though: In set theory we use an *axiom* of infinity to ensure the existence of a set containing all the sets in such a sequence; similarly for other theories we might use in the background, e.g., an intensional logic. And as we have learned from Gödel, via his second incompleteness theorem, there are deep problems with proving the consistency of such an axiom, at least by more elementary means. Having said that, it is not clear that Dedekind or a contemporary logical structuralist needs consistency to be *provable*. Arguably, all that is needed is that the axioms in question *are* consistent – and virtually nobody doubts that fact today.

15. CONCLUSION

It is not hard to see that Dedekind is a *methodological structuralist*, as defined initially in this paper. In addition, I have argued for an interpretation of him as a *logical structuralist*. Seen as such, he holds certain metaphysical views about the identity and nature of the natural numbers, especially about their determinateness, that go beyond merely methodological, mathematical concerns. Crucial for my interpretation has been a new understanding of Dedekind's notions of *abstraction* and *free creation*. I understand them in a logical way, and reject a crude psychologistic reading of Dedekind's corresponding remarks.

Overall, my discussion of Dedekind's position has focussed on logical and metaphysical aspects – intentionally so. It has sometimes been claimed that Dedekind did not hold metaphysical views, or at least that he was not concerned with them in any serious way.[45] If my interpretation is correct, this is not true. Rather, his main mathematical and logical insights are, in his own view, *intimately linked* with metaphysical issues. This becomes clear if we take seriously his notions of abstraction and free creation.

Having said that, I agree that it is possible to *separate* the logical and mathematical contributions Dedekind made from his metaphysical conclusions. One way to do so is by considering his definition of a simply infinite system as an early expression of the Dedekind-Peano Axioms, and by then treating these axioms in a contemporary formal or model-theoretic way. If we do that, Dedekind's notions of abstraction and free creation either drop out completely or they are interpreted along set-theoretic structuralist lines. Indeed, along such lines one can put aside all metaphysical questions, e.g., if one treats set theory in a purely inner-mathematical, formal way.

Why then *should* we take seriously Dedekind's remarks about abstraction and free creation? Why not focus exclusively on his mathematical and logical contributions? My answer to this question is twofold: First, it is only if we take Dedekindian abstraction and free creation seriously that we get an accurate, full understanding of his views. Second, taking them seriously provides us with a way of justifying something that many mathematicians do, although it is not necessary for mathematical purposes. Namely, they often talk about "the natural numbers" as if this phrase refers to a definite, unique mathematical system; likewise, they use the numerals '1', '2', '3', ... as if they refer to definite, unique objects. How should we understand such talk? It is certainly possible to reinterpret it along either set-theoretic structuralist or modal structuralist lines. But if we do so, we re-analyze what mathematicians say in an indirect, non-literal way. What Dedekind does is to provide us with a way of taking it *literally*. That is to say, his notions of abstraction and free creation, coupled with his principle of determinateness for mathematical objects, provide us with an interesting and subtle justification for talking about mathematical objects – the best such justification of which I know.

A final clarification about my interpretation of Dedekind, or about its intended scope. In this paper I have focussed on Dedekind's essay "Was sind und was sollen die Zahlen?". My main mathematical example has, therefore, been the natural numbers. What about Dedekind's other writings; and what about other mathematical objects, e.g., the real numbers? As my brief discussion of Hilbert's "Über den Zahlbegriff" shows, I do think that Dedekind's treatment of the natural numbers can be transferred directly to the real numbers. Indeed, I think this is exactly what Hilbert does. It is even arguable that all the main mathematical ingredients for this treatment are already present in Dedekind's own "Stetigkeit und irrationale Zahlen". Moreover, the same treatment can be transferred further: to any mathematical theory that is consistent and categorical.[46]

Then again, Dedekind wrote his essay on the real numbers a few years before that on the natural numbers, and not all crucial ingredients of the later essay are present in the former, at least not explicitly. For example, there is no special notion in the essay on the reals that corresponds to the notion of a simply infinite system, such as our current notion of a complete ordered field. A second, perhaps more important ingredient missing is an explicit theorem and proof for categoricity. Third, while Dedekind does talk about "free creation" in connection with the real numbers, his most illuminating discussion of it and, especially, of the notion of "abstraction" occurs in "Was sind und was sollen die Zahlen?" The latter essay remains, then, the classic statement of Dedekind's logical structuralism.

ACKNOWLEDGMENTS

The present paper is closely related to (Reck and Price 2000), in some respects also to (Awodey and Reck 2001). My approach is strongly influenced by (Tait 1986) and (Stein 1988). Earlier versions, or precursors, of the paper were presented at the University of California at Irvine, May 1996, and the University of Chicago, June 1996. At each occasions I received valuable feedback from the audience. More recently, I have also benefitted from conversation about Dedekind and structuralism with Pierre Keller, Dirk Schlimm, and Wilfried Sieg, as well as from helpful comments by two anonymous referees. Finally, I would like to thank the Niedersächsische Staats- und Universitätsbibliothek Göttingen, in particular Dr. H. Rohlfing, for making available to me unpublished material from Dedekind's Nachlaß.

NOTES

1 For a more detailed comparison of them see (Reck and Price 2000); compare also (Parsons 1990) and (Hellman 2001).

2 In (Reck and Price 2000) the more inclusive rubrics of "methodological structuralism", "relativist structuralism", "universalist structuralism", and "pattern structuralism" are used instead. Compare (Shapiro 1997) and (Hellman 2001) for related taxonomies.

[3] For more on what is meant by "general" and "conceptual" here, as opposed to "computational" and "particularist", see (Stein 1988). For more on "formal axiomatics", see (Awodey and Reck 2001).

[4] For a more detailed characterization of modal structuralism, including a clearer distinction between its universal and modal aspects, see again (Reck and Price 2000).

[5] In what follows, I will refer to definitions and theorems in (Dedekind 1888b) by the numbers Dedekind gave them. Quotations in English, with page numbers, will be from (Dedekind 1963), although I will sometimes amend the translations. Similarly later for Dedekind (1872).

[6] (Dedekind 1988b), Definitions 44, 26, and 71, respectively. Dedekind first defines the notion of "chain" more generally, in Definition 37.

[7] I have changed the notation in this definition slightly so as to make it easier to follow for a contemporary reader.

[8] (Peano 1889), p. 103 (of the English translation).

[9] Dedekind most likely found out about this antinomy from Cantor; see fn. 17 in (Parsons 1990). He definitely took it very seriously. In particular, he resisted the publication of a new edition of "Was sind und was sollen die Zahlen?" because of it. Compare also the following report by the set-theorist Felix Bernstein, written after a conversation he had with Dedekind in 1897: "Dedekind hadn't arrived at a final position [about the antinomy] then; and he told me that in considering it he had almost arrived at doubts about whether human thinking was fully rational" (Dedekind 1932, p. 449, my translation).

[10] For more on these two issues see sections 7 and 9–10, respectively.

[11] Theorem 33 establishes the transitivity of "similarity": "If *R* and *S* are similar systems, then every system *Q* similar to *R* is also similar to *S*" (ibid., p. 55).

[12] In fact, by devoting two explicit theorems, with proofs, to this fact Dedekind is far ahead of his time; compare (Awodey and Reck 2001).

[13] Concerning Dedekind's mathematical works more generally, compare (Stein 1988) and (Schlimm 2000).

[14] See section 11.

[15] We will come back to this issue in a later section (Section 12), in connection with additional evidence from Dedekind's Nachlaß. It will turn out that more is going on beneath the surface than the argument just given suggests; thus my repeated use of "seems" etc. in this passage.

[16] For more on Russell in connection with structuralism, in particular universalist structuralism, see again (Reck and Price 2000).

[17] Brief discussions of these constructions can be found in Dedekind's Nachlaß; compare Appendix 3 in (Schlimm 2000), pp. 101–106. It is an interesting question (to which I don't know the answer) when and by whom they were introduced for the first time. In the passage quoted Dedekind treats them as if they were well-known at the time.

[18] If we compare Dedekind's writings from the 1850s and 60s to those from the 1870s and 80s, one can, indeed, find significant changes; compare, e.g., what he says about the natural numbers in (Dedekind 1854). Thus there is reason to wonder about further changes later as well.

[19] It is interesting to compare what Dedekind says about the natural and real numbers to his theory of ideals. In that case, too, he employs set-theoretic constructions, even in the sense that ideals, like cuts, are constructed as infinite sets of numbers. However, Dedekind never, as far as I know, suggests "creating" new numbers, or new objects, corresponding to these ideals. Howard Stein, in conversation, has suggested to me some plausible reasons for the difference in Dedekind's approach: (i) In ideal theory there is not just one canonical extension of the old domain, but many different and equally important ones; (ii) we cannot extend all the numerical operations (in a straightforward way); (iii) the set-theoretic properties of ideals play a rather direct role in this case, or at least more so than in the cases of the natural and real numbers.

[20] For many contemporary readers this argument will be familiar from Paul Benacerraf's article "What numbers could not be" (Benacerraf 1965). However, Dedekind goes on to draw very different conclusions from it than Benacerraf, as we will see soon. Actually, the same argument – that classes cannot be numbers because they have the wrong properties – is also mentioned in (Russell 1903), p. 115, and attributed to Giuseppe Peano.

[21] A slightly different, though not unrelated, interpretation of Dedekind's argument has been suggested to me by Howard Stein: Perhaps Dedekind's refusal to identify the real numbers with cuts has to do with cuts being of the *wrong type* (in something like Russell's sense); similarly in the case of the natural numbers. This suggestion gives another sense to the idea of being, or not being, "simple" and "orderly". However, apart from the remark about "uniformity" at the beginning of the passage above, which can perhaps be seen as a hint in this direction, I do not know of any evidence for type-theoretic concerns in Dedekind's writings.

[22] These two theses constitute a particular and relatively crude version of psychologism. It is this kind of position that is often attributed to Dedekind, I believe. For more general discussions of psychologistic views, without direct reference to Dedekind, compare (Rath 1994) and (Kusch 1995).

[23] See (Kitcher 1986); compare also (McCarty 1995). Of course, whether to understand Kant's transcendental idealism as a kind of psychologism or not is controversial in itself. By talking about a "sophisticated variant", I mean to suggest that Kant's position is not a crude version of psychologism, but to leave the precise interpretation open.

[24] For more on the later issue see the next section (Section 8).

[25] Pursuing the general Kantian theme further, Dedekind's rejection of intuition and his replacement of it by logic actually leads in the direction of Neo-Kantianism, especially Ernst Cassirer's views; see, e.g., (Cassirer 1910). Compare here (Friedman 2000), chapters 3 and 6. I plan to explore the connections between Dedekind's and Cassirer's views on mathematics further in a future publication. (I am grateful to Pierre Keller for first pointing me in this direction).

[26] See (Stein 1988) and (Tait 1997). The discussion that follows is strongly influenced by both Stein and Tait, also via conversations.

[27] In the original German: "Die Zahlen sind unabhängig von Raum und Zeit, sie entstehen unmittelbar aus den Gesetzen des menschlichen Denkens. (Dirichlet, *Vorlesungen über Zahlentheorie*, dritte Auflage, 1879, §163. Anmerkung auf Seite 470)".

[28] The term "conceptual possibility" is from (Stein 1988), p. 252.

[29] For further discussion of this issue, including a precise definition of "semantic completeness", see again (Awodey and Reck 2001).

[30] As an anonymous referee has reminded me, one should really consider this essay not in isolation, but in connection with Hilbert's other early works, especially *Grundlagen der Geometrie* (1899). I agree, but think that doing so would only confirm the interpretation to be given in this section. I plan to discuss Hilbert's writings more generally, in connection with structuralism, in future work. Compare also (Sieg 2002) in this connection.

31 For more on Hilbert's axioms, including a discussion of differences to contemporary versions and a comparison to Dedekind's approach to the real numbers, see (Awodey and Reck 2001).

32 For further discussion of this issue from a Hilbertian point of view see (Bernays 1950).

33 Actually, while Hilbert hints at the categoricity of his axioms for the real numbers, he is much less explicit about this aspect than Dedekind was in connection with the natural numbers; compare again (Awodey and Reck 2001).

34 An exception is (Sieg 2002). In my brief remarks about Hilbert in this paper I see myself as agreeing largely with Sieg's interpretation of him as a "reductive structuralist".

35 In this section I am am trying to elaborate on ideas first presented in (Tait 1997), especially on the notion of "Dedekind abstraction".

36 This last point will have to be finessed somewhat in the end; see Section 13.

37 See here (Shapiro 1997), pp. 40–41. Of course, such a metaphysical reading of Plato is not universally shared; compare, e.g., the corresponding asides in (Tait 1986).

38 This objection was already raised by Russell, in a passage quoted earlier: "What Dedekind presents to us is not the numbers, but *any* progression: what he says is true of *all* progressions alike, and his demonstrations nowhere – not even where he comes to cardinals – involve any property distinguishing numbers from other progressions". (Russell 1903, p. 249, emphasis added).

39 In the original German: "Da durch diese Abstraktion die ursprünglich vorliegenden Elemente *n* von *N* (und folglich auch *N* in ein neues abstraktes System $\mathfrak{N}$) in neue Elemente $\mathfrak{n}$, nämlich die Zahlen, umgewandelt werden, so kann man mit Recht sagen, daß die Zahlen ihr Dasein einem freien Schöpfungsakt des menschlichen Geistes verdanken. Für die Ausdrucksweise ist es aber bequemer, von den Zahlen wie von den ursprünglichen Elementen des Systems *N* zu sprechen und den Übergang von *N* zu $\mathfrak{N}$, welcher selbst eine ähnliche Abbildung ist, außer Acht zu lassen, wodurch, wie man sich mit Hilfe der Sätze über Definitionen durch Rekursion [...] überzeugt, nichts Wesentliches geändert, auch nichts auf unerlaubte Weise ausgelassen wird." (Dedekind clearly uses two different kinds of letters for *N* and $\mathfrak{N}$, as well as for *n* and $\mathfrak{n}$, in the original).

40 What I mean by "abstract" here is basically: not located in space and time, and causally inert (where the latter means that a corresponding entity doesn't have the potential to act or be acted on causally, except perhaps indirectly by being thought about).

41 This challenge can be seen to have been raised by Paul Benacerraf, among others, in remarks such as the following (in connection with a view very similar to Dedekind's): "[N]umbers are not objects at all, because in giving the properties (that is necessary and sufficient) of numbers you merely characterize an *abstract structure* – and the distinction lies in the fact that the 'elements' of a structure have no properties other than those relating them to other 'elements' of the same structure. [...] That a system of objects exhibits the structure of the integers implies that the elements of that system have some properties not dependent on structure. (Benacerraf 1965, p. 291, original emphasis).

42 While I consider the response to our second objection just given to shift the burden of proof to the critic, I should add the following: Both (Hellman 2001) and (Keränen 2001) contain recent attempts to elaborate further what is problematic about the abstract objects appealed to by structuralists à la Dedekind and Shapiro. (Both of them focus on Shapiro.) I plan to examine their challenges – especially whether they really apply to Dedekind, also whether and how they go beyond Russell's and Benacerraf's – in a future publication. The fact that I cannot do so here, for lack of space and time, is partly what makes this paper only a *partial* defense of Dedekind's views. Compare also fn. 46.

43 Here I am interpreting Dedekind as implicitly relying on something like the notion of "structure equivalence", as opposed to strict "isomorphism", in determining the identity of the natural numbers; see (Shapiro 1997), p. 91, and the corresponding discussion in (Reck and Price 2000), pp. 368–369.

44 See (Awodey and Reck 2001) for further discussion of this issue.

45 Compare (Stein 1998), p. 247: "[Dedekind] is not concerned, unlike Frege, to identify numbers as particular 'objects' or 'entities'; he is quite free of the preoccupation with 'ontology' that so dominated Frege, and has so fascinated later philosophers." (In practically all other respects I agree with Stein's article, and I am strongly influenced by him more generally, as indicated in earlier footnotes).

46 Recently it has been argued that the existence of non-trivial automorphisms for the models of some categorical theories, such as the theory of the complex numbers or, more severely, that of the Euclidean plane, causes problems for a structuralist along the lines of Dedekind and Shapiro; compare (Burgess 1999) and (Keränen 2001) (who both focus on Shapiro). I am not sure about the real force of these arguments, nor about whether and how they affect the line of thought developed in this paper. Note, in addition, the following: In the cases Dedekind is most interested in, the natural and the real numbers, we are dealing with theories that are not just categorical, but *uniquely categorical*, i.e., between any two models there exists a unique isomorphism (so that there are no non-trivial automorphisms); compare (Awodey and Reck 2001), section 6. Insofar as arguments such as those by Burgess and Keränen do have force, it should suffice, then, to restrict oneself to such theories. Nevertheless, this issue deserves more attention than I can give it here. Thus what I say towards the end of fn. 42 applies again.

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