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## FREGE, NATURAL NUMBERS, AND ARITHMETIC'S UMBILICAL CORD\*

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**Abstract:** A central part of Frege's logicism is his reconstruction of the natural numbers as equivalence classes of equinumerous concepts or classes. In this paper, I examine the relationship of this reconstruction both to earlier views, from Mill all the way back to Plato, and to later formalist and structuralist views; I thus situate Frege within what may be called the "rise of pure mathematics" in the nineteenth century. Doing so allows us to acknowledge

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\* Both with respect to its general perspective and various details, this paper is strongly influenced by W.W. Tait (not only via publications, but also conversations and unpublished manuscripts). Parts of the paper were presented in talks at the University of California at Santa Barbara, December 2000, and at California Polytechnic State University San Luis Obispo, May 2002. I would like to thank Kevin Falvey and Francisco Flores for the respective invitations, and the members of the two audiences for their interest and comments. More recently, I have received very valuable feedback from Pierre Keller, Marco Ruffino, Kai Wehmeier, and especially Mike Price. As I have not followed all of their advice, and insisted on some of my own twists to the account, the remaining problems should, as usual, be attributed to me.

continuities between Frege's and other approaches, but also to understand better the motivation and the significance of his innovations, as well as their limits.

Key-words: Arithmetic. Concept. Equivalence class. Extension. Logicism. Plurality.

Frege made important, indeed revolutionary, contributions to logic, the philosophy of logic, and the philosophy of thought and language. But his primary concern was with the philosophy of mathematics, in particular the foundations of arithmetic. At the center of his work in this area lies a reconceptualization of the natural numbers. In the present paper I want to shed new light on this reconceptualization by doing two related things: first, by making explicit the relation of Frege's conception of numbers to other such conceptions, especially earlier ones, from Mill all the way back to Plato; and second, by locating Frege's conception within what may be called the "rise of pure mathematics", during the nineteenth and the early twentieth centuries, leading up to contemporary formalist and structuralist views on the subject.

In his own writings, Frege sometimes makes it appear as if his new conception of numbers constituted a radical break with all earlier conceptions. My first main goal in this paper will be to show that this appearance is misleading and partly wrong. Instead, one can see Frege's position as a natural extension of earlier ideas and developments, at least in certain respects. At the same time, in other respects his position does indeed involve substantive, significant changes. My second goal will be to clarify what these changes consist in and why they are significant. As a third goal, I will try to determine how far Frege's changes go, or what their limits are, particularly in comparison to later positions. Overall, I will sketch

an approach to Frege's conception that, as far as I am aware, has not been explored much in the literature.<sup>1</sup>

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Frege published most of his works during the last part of the nineteenth century. That century was not only a time of explosive growth in mathematics, but also one during which deep changes took place with respect to how people conceived of mathematics, especially its conceptual foundations, and even its subject matter.<sup>2</sup> In order to direct attention to what is crucial for me in this connection, I will start by discussing briefly the more familiar case of geometry, in particular a somewhat dramatic characterization of what happened in it provided by the historian of mathematics Hans Freudenthal. In Freudenthal's memorable phrase, a main result of developments in the nineteenth century is that "the umbilical cord between geometry and reality has been cut" (Freudenthal, 1957, p. 111, my translation; to be quoted more fully below).<sup>3</sup>

<sup>1</sup> Much recent work on Frege's philosophy of arithmetic is directed towards technical issues, i.e., the exploration of what follows formally from his basic principles, or from modifications of them; cf. many of the articles in Demopoulos, 1995. Other work concerns metaphysical aspects of his conception of numbers, especially its "platonist" character; cf. Reck, 1997 and Reck, 2000 for my own contributions to those debates. The present paper focuses on conceptual and historical aspects, in a way that thoroughly intertwines the two. For papers with a similar focus, compare Tait, 1997, partly also Wilson, 1992.

<sup>2</sup> With respect to these changes, Howard Stein has talked about a "second birth" of mathematics in the nineteenth century (Stein, 1988). Jeremy Gray, referring to the same period, has talked about a "revolution in mathematical ontology" (Gray, 1992).

<sup>3</sup> In the original German: "[D]amit ist die Nabelschnur zwischen Realität und Geometrie durchgeschnitten". Compare also the later, shorter English

What is the “umbilical cord” Freudenthal is talking about, and in which sense has it been cut? In order to answer these questions, we need to contrast two different conceptions of geometry that can be located, more or less, at the beginning and at the end of the corresponding developments.<sup>4</sup> The conception at the beginning is this: Geometry (here in the sense of three-dimensional Euclidean geometry) was born, in antiquity, as the systematic study of spatial relations between various objects. What “spatial” refers to here is ordinary empirical or physical space, the space in which we compare the distance between cities, measure the areas of patches of land, etc. To be sure, when geometers study points, lines, planes, etc. in space, they conceive of them in an idealized way; they conceive of, say, a straight line as perfectly straight and as having no breadth or width, only length. This has the consequence that physical objects or phenomena, such as light rays, exemplify such straight lines always only approximately. Nevertheless, we think of the ideal points, lines, planes, etc. as located, or locatable, in ordinary space. From such a perspective, the axioms which, since the time of Euclid (ca. 300 BC), have formed the starting point of geometry are taken to be evident, basic truths about space, or about

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version Freudenthal, 1962, p. 618, in which “*Nabelschnur*” is not translated as “umbilical cord”, but simply and less memorably as “bond”.

<sup>4</sup> I say “more or less” since some of the crucial ingredients of the conception of geometry I locate in the late nineteenth and the twentieth centuries can, arguably, already be found much earlier, e.g., in Plato’s dialogues; compare the discussion of mathematics as *a priori* reasoning from first principles in Tait, 2002. If so, then both of the conceptions at issue have been around for a while, at least in rudimentary, partly implicit forms. Having said that, the second conception, unlike the first, was articulated fully only in the late nineteenth century, in the works of Hilbert and others; and it became the dominant view only in the twentieth century. I take that to be Freudenthal’s main point.

the properties and relations of idealized points, lines, planes, etc. in it. This is what I will call the “practical” or “applied conception” of geometry. One may also call it the “naïve” conception, in the sense that many people not trained as mathematicians or philosophers of mathematics will assume it as a matter of course, even today.

With this first conception, geometry is firmly attached to physical reality – the umbilical cord between them is still in place. Things look very different when we consider the conception of geometry presented in David Hilbert’s *Grundlagen der Geometrie* (first published in 1899). Along Hilbert’s lines, we start again with a set of geometric axioms. But these axioms are no longer considered to be evident truths about ordinary space; rather, they are taken to form the definition or characterization of a certain abstract structure. Crucially, we can now think of various different sets of objects and relations on them as exemplifying this structure. To use terminology from twentieth-century logic, we can consider the language of geometry as a formal, uninterpreted language, and we can then study various models for the axioms formulated in that language. Moreover, we can construct such models purely mathematically, within set theory, thus in complete separation from empirical considerations. As a consequence, when we now talk about “points”, “lines”, “planes”, etc., we refer to objects in such models, typically pure sets. One advantage of this more abstract, formal, or logical conception is, of course, that we can treat not only the axioms of Euclidean geometry along such lines, but also the axioms of various non-Euclidean geometries. The latter, too, can be seen as characterizing correspondingly different geometric structures.

Reflecting on this second conception of geometry, Hans Freudenthal comments (with reference to the opening lines of Hilbert’s *Grundlagen*):

"We think of three different things..." – thus the umbilical cord between reality and geometry has been cut. Geometry has become pure mathematics, and the question whether and how it can be applied to reality is answered just as in any other branch of mathematics. The axioms are no longer evident truths, indeed it doesn't even make sense any more to ask about their truth. (Freudenthal, 1957, p. 111, my translation)

The crucial "cut of the umbilical cord" is to distinguish sharply between, on the one hand, geometry as the purely formal or logical study of what follows from the geometric axioms, where these axioms are considered to characterize an abstract structure or a certain kind of relational systems, and, on the other hand, the application of geometry to physical reality, seen as an independent activity. Thus the former, "pure geometry", is taken to be part of the general, abstract study of diverse relational systems, with no immediate or necessary connection to exemplifications in physical reality. It is this new conception of geometry that, soon afterwards, leads philosophers such as Rudolf Carnap and Hans Reichenbach to distinguish between "formal" or "mathematical space" (or better, "spaces"), on the one hand, and "physical" or "empirical space", on the other.<sup>5</sup> The first is what we study in pure mathematics, the second what we study in physics.

Clearly the development of geometry, especially in the nineteenth century, had its own dynamic and was different from that of arithmetic in a number of important respects. In particular, the rise of various non-Euclidean geometries, differential geometries, and of projective geometry played an important role in the

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<sup>5</sup> See Carnap, 1922, chapters 1-2, and Reichenbach, 1951, chapter 8. In addition to "formal" and "physical space", Carnap considers what he calls "intuitive space" in his early work. This is an attempt to make room for Kant's views about space that he gives up later.

reconceptualization of Euclidean geometry just mentioned, and these developments have no obvious or full analogues in the case of arithmetic.<sup>6</sup> Nevertheless, parallel steps towards conceiving of arithmetic in a formal or logical, thus "pure", way were taken as well. These steps are what I want to consider now (eventually so as to locate Frege with respect to them). Doing so will again involve describing a practical or naïve conception, on the one hand, and a formal or logical conception, on the other. And once more, the latter will, but the former will not, involve a separation of pure arithmetic from its applications.

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If geometry deals with the systematic relations between points, lines, planes, etc., then arithmetic deals with the natural numbers, relations between them (their ordering etc.), and various functions or operations performed on them (such as addition and multiplication). And if we try to think of a naïve conception of arithmetic, this may at first bring to mind the following simple formalist position: What are numbers, or what is arithmetic about? Well, simply numerals, i.e., concrete figures such as "1", "2", "3", etc., used to count, calculate, etc. Such a naïve formalist conception has certainly played a role, both in philosophy and in popular consciousness.<sup>7</sup> Yet it is not what corresponds to the practical conception of geometry considered above, at least for my purposes.

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<sup>6</sup> Perhaps Hamilton's theory of quaternions and related developments in algebra, Dedekind's, Cantor's, and others' new treatments of the real numbers, and Cantor's introduction of transfinite cardinal and ordinal numbers can be seen as partial analogues. (I will come back to both Dedekind's and Cantor's corresponding conceptions of the natural numbers later.)

<sup>7</sup> As to popularizations of mathematics, this conception underlies books such as Ifrah, 2000.

In order to arrive at a corresponding conception, we need to consider the use of numerals in practical applications, especially in the comparison of “numbers of things” with respect to their “size” (i.e., the use of natural numbers as cardinal numbers).

What is meant by a “number of things” here? It is, at least initially and roughly, the following: a concrete group or collection of things, e.g., a heap of stones, a flock of sheep, or a pack of cards. Another term often used in connection with such groups or collections is “plurality”. A plurality, in this sense, is something that can be “counted” or “numbered”. More carefully, it can be numbered relative to a partition into parts, or relative to a corresponding “choice of unit”.<sup>8</sup> Now, it seems that in practically all cultures and all languages there exist words or signs for talking about such pluralities, including for comparing them in size; in English we talk about “three stones”, “ten sheep”, etc. Moreover, the corresponding conception of “number of things” was explicitly assumed to be fundamental for arithmetic already in classical Greek mathematics, as the use of the term “*arithmos*” indicates. An *arithmos* – usually just translated as “number” – is exactly a plurality in the sense introduced, i.e., a (definite, finite) number of things.<sup>9</sup>

<sup>8</sup> The phrase “choice of unit” is clearest in the case of, say, a line segment that is partitioned into a number of parts relative to a “unit length”. This kind of example leads in the direction of “magnitudes”, thus eventually to the rational and real numbers. As I want to restrict myself to the natural numbers in this paper, I will focus on simpler, discrete kind of examples in what follows. (I will come back to the notion of “choice of unit” later, in connection with Frege.)

<sup>9</sup> In my brief discussions of classical Greek thought in this paper, especially concerning the notions of plurality, *arithmos*, and their relation to nineteenth century developments, I am strongly influenced by W.W. Tait; see Tait, 1997 and 2002. In addition, compare Klein, 1968, chapter 6ff, and Roche, 1998, chapter 1.

As an aside that will play a small role later in the paper, we can note right away a peculiar consequence of thinking of numbers as pluralities. Namely, it is natural to assume that it only makes sense to think of a plurality if we are dealing with several things – things in the plural. But if we assume that, the conclusion is that 1 is not really a number, much less 0 (not to speak of negative numbers etc.). This becomes most compelling if we conceive of numbering in terms of partitioning things into parts, as indicated above; since it hardly makes sense to partition something into one part (itself?), or to partition something (nothing?) into no parts at all. For this very reason, 0 and 1 were often not considered to be numbers, from Greek mathematics onwards (similarly again for negative and other, more exotic numbers).

Actually, what I just said about the use of the term “*arithmos*” in classical Greek mathematics needs to be refined. It is true that historians of mathematics such as Jacob Klein have claimed that the term “never means anything other than a definite number of definite things” in Greek thought (Klein, 1968, p. 7). Moreover, this may be accurate as far as practical applications of arithmetic in classical Greece are concerned. But at least if we also take into account Greek philosophers and their reflections, the issue starts to look more complex. For clarification, it will help to introduce another term often used in this connection, interchangeably with “plurality”, namely “multitude”. More specifically, let us consider the use of the phrase “multitude of units”. This phrase occurs, among others, in the influential definition of numbers in Euclid’s *Elements*, Book VII, where we can read: “A number [*arithmos*] is a multitude composed of units” (Euclid, 1956, p. 277). The relevant question now is: what are the “units” in this connection?

If by “multitude”, thus by “*arithmos*”, we mean a “definite number of definite things”, say a flock of ten sheep, then each of the sheep will presumably be considered a “unit”. However, an

alternative conception can already be found in Plato's works, e.g., in the following passage from the *Philebus*:

The ordinary arithmetician, surely, operates with unequal units; his 'two' may be two armies or two cows or two anythings from the smallest thing in the world to the biggest, while the philosopher will have nothing to do with him, unless he consents to make every single instance of a unit precisely equal to every other of its infinite number of instances. (Plato, 1989, 56d-e)

It seems that, according to "the philosopher", we have to abstract away from the differences between concrete units, such as individual sheep, so as to arrive at pure, indistinguishable units; and if we do that, an *arithmos* turns into a multitude of such abstract units. As Plato also puts it, depending on whether we think of "units" in a more concrete or a more abstract manner we are dealing with "physical numbers", on the one hand, and "abstract numbers", on the other.<sup>10</sup> Because of such distinctions, the phrase "multitude of units", as used by Euclid and others, is ambiguous.<sup>11</sup>

<sup>10</sup> For Plato's more general views about arithmetic, compare also the passages from the *Republic*, *Theaetetus*, and other texts quoted in Klein, 1968, Roche, 1998, and Tait, 2002. Note that, with his more abstract understanding of "unit" and "multitude of units", Plato goes beyond our practical, naïve conception of number, perhaps even (as suggested to me by Pierre Keller) in the direction of a structuralist conception. (Compare here fns. 4, 12, and 16; see also my remarks below about Cantor's late nineteenth-century conception of the natural numbers.)

<sup>11</sup> Besides Euclid and Plato, one can find characterizations of the natural numbers as "multitudes of units" also in, e.g., Aristotle (384-322 BC), Boethius (ca. 480-524 AD), and John of Sacrobosco (died ca. 1244-66), so all through Hellenistic Greece and into the European Middle Ages; cf. again chapter 1 of Roche, 1998 for references. The same basic idea occurs again in modern philosophers such as David Hume (1711-1776), Immanuel Kant (1724-1804) (at least to some degree), and J.S. Mill (1806-73), as we will see

How exactly to understand the Platonic idea of "abstract numbers", or of a "multitude of abstract units", has been controversial from early on (at least since Aristotle's works); and it is certainly in need of clarification.<sup>12</sup> But let me put aside the corresponding abstract conception of numbers until later, and explore a bit more the concrete, practical conception that works with "physical numbers". Conceiving of numbers as such leads naturally to a traditional view about the semantics of arithmetic terms and sentences. Consider the following question: What is it that a numeral such as "2", or a word such as "two", refers to? That is to say, what is it that we talk about when we use an equation like " $2 + 3 = 5$ ", or a corresponding sentence in words? The traditional view I have in mind answers that the numerals "2", "3", etc. are "common names".<sup>13</sup> This means that "2", say, is a term used to refer to all multitudes that consist of exactly two things. Consequently, when we assert that " $2 + 3 = 5$ " we are making a general statement of the following form: For any multitude consisting of two things and any multitude consisting of three (other) things, if we combine them we arrive at a multitude consisting of five things. Finally, given the "physical" conception of multitudes we are currently assuming this

soon. In each case it is an interesting question, I think, whether what is meant by "multitudes of units" is Plato's "physical numbers", his "abstract numbers", or some amalgam of the two.

<sup>12</sup> One question in this connection is how to understand the notion of "multitude" (cf. my corresponding discussion later on in this paper). A second question is how to conceive of Plato's "abstract units": as new abstract objects, either of a spatio-temporal nature (comparable to dimensionless points in space, say) or of a non-spatio-temporal nature (cf. again below); or perhaps as familiar concrete objects conceived of in some abstract way. Thirdly, there is the more general, big question of how to relate Plato's views about such notions to his theory of forms.

<sup>13</sup> Aristotle, 1984, 224<sup>a</sup>3-16, is a *locus classicus* for this view.

leads to the more general conclusion: When we use arithmetic terms and sentences, we are always directly talking about things in the physical world. In other words, the umbilical cord between arithmetic and physical reality is still firmly in place – as much so as in the case of our naïve conception of geometry above.

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As in the case of geometry, the practical view about numbers just considered is “naïve” in the sense that many, perhaps most people will find it natural and congenial, especially if they have not been trained as mathematicians or philosophers of mathematics.<sup>14</sup> But even various philosophers, all the way into the nineteenth century, have relied on the same basic conception. A particularly clear example is John Stuart Mill. In his *A System of Logic*, Mill asserts that “ten must mean ten bodies, or ten seconds, or ten beatings of the heart” (Mill, 1950, p. 163 ff.). More generally, he talks about “aggregates”, “agglomerations”, or “parcels of objects” in a sense close to the concrete pluralities or multitudes introduced above. Finally, he states explicitly that a sentence such as “ $2 + 3 = 5$ ” expresses a general truth, again in the way indicated above.

The example of Mill also allows me to make explicit an aspect of the conception of numbers as concrete pluralities that has remained largely implicit so far. Up to this point, our focus has been on the nature of numbers, not on the nature of the corresponding numerical operations, such as addition and multiplication; but

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<sup>14</sup> As far as I know (from my sister, who works as an elementary school teacher, and her textbooks) the way in which most people learn about the natural numbers in elementary school is, not unreasonably, still largely informed by the pluralities conception, i.e., by looking at various concrete “numbers of things”, comparing them in size, etc. Many people never have reason to go beyond such a conception later on.

clearly the latter is important as well. Consider addition. Above I stated, briefly and vaguely, that “ $2 + 3 = 5$ ” involves a combination of two multitudes into a new multitude. What exactly is meant by “combination” here? Mill, for one, is explicit and straightforward on this issue. For him, such combination involves physical operations, especially operations of “separation and rearrangement” (*ibid.*, p. 166). In other words, not only are numbers of things, conceived of as “aggregates” or “parcels”, concrete entities for him – as concrete as the things they consist of, e.g., physical bodies or beatings of the heart – but addition and multiplication amount to equally concrete operations in the physical world.<sup>15</sup> Overall it is understandable, then, how Mill is led to conceiving of statements such as “ $2 + 3 = 5$ ” as empirical statements about the physical world.

Mill’s very concrete and applied way of interpreting arithmetic, including arithmetic operations, is clearly extreme and somewhat heavy-handed. Subtler variants of the same basic ideas can, however, be found in a number of other writers as well. An interesting case is provided by Kant’s views. How exactly Kant thinks about the nature of numbers is not easy to determine, partly because he says more about numerical operations than about numbers, partly because this leads us right into the intricacies of his transcendental aesthetics and analytic (his notion of schema etc.). But consider remarks such as the following about the addition of two numbers, here 5 and 7:

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<sup>15</sup> I assume that Mill takes “+” to be a general term as well, i.e., to refer to all relevant (?) physical operations, but I am not sure. Sometimes he seems to restrict himself to very particular physical movements, involving the creation of specific spatial configurations.

For starting with the number 7, and for the number 5 calling in the aid of the fingers of my hand as intuition, I now add *one by one* to the number 7 the units which I previously took together to form the number 5, and with the aid of the figure [the hand] see the number 12 come into being. (Kant, 1965, B 15-16, original emphasis)

As this passage shows, Kant (following Leibniz in this respect) reduces the operation of “+” to that of “+1”, in familiar recursive fashion. More importantly for our purposes, he talks about “adding units”. It is not immediately clear whether he is appealing to concrete or to abstract units in this connection, or perhaps both.<sup>16</sup> In any case, the passage suggests that built into his position is again the notion of “multitudes of units” (or perhaps better: the “generation” of such multitudes). Kant then combines it with his theory of intuition, including “pure intuition” (in terms of the relevant generation of multitudes in space and time). And as a result, arithmetic statements and knowledge turn out not to be empirical, but *apriori*, since anchored in our intuitive forms of space and time. Note that, in this more subtle sense, the umbilical cord between arithmetic and spatio-temporal reality is preserved in Kant as well.

Let me add one more clarification about the conceptions of numbers considered so far. Clearly, conceiving of them as pluralities

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<sup>16</sup> That Kant means abstract, indistinguishable units, at least also, seems to be suggested by passages such as this: “[T]he pure *schema* of magnitude (*quantitatus*), as a concept of the understanding, is *number*, a representation which comprises the successive addition of *homogeneous units*” (Kant, 1956, A 142/B 182, last emphasis added). Note that, as in the case of Plato (cf. footnotes 4, 10, and 12 above), it may be possible to interpret Kant’s conception of numbers as having a structuralist aspect, in this case one closely tied to his theory of space and time. I cannot explore this interpretive possibility (also suggested to me by Pierre Keller) in the present paper, but hope to be able to do so elsewhere.

or multitudes of units was very widespread in the history of thought, from Plato all the way to Mill. Its main alternative was probably the simple formalist view – numbers as numerals – introduced briefly above. Now, it is possible to see these two conceptions as reconcilable with each other. To do so, consider the collection of numerals from “1” to “5”, say, as itself a multitude of concrete units. This multitude can then be used, as a “tally”, to measure other multitudes, e.g., the one consisting of the planets in the Solar System from Mercury to Saturn. The measuring or tallying here is supposed to consist of assigning one numeral to each planet, in such a way that all the numerals and all the planets are used up, i.e., in establishing a concrete bijection between the two pluralities. Instead of numerals we can, and often do, use collections of other items as tallies as well, e.g., fingers on a hand (cf. the quote from Kant above) or beads on an abacus. Furthermore, we can compare two multitudes in size directly, without the use of an intermediate tally, by means of concrete injections or bijections. But collections of numerals are, in various familiar ways, especially convenient for measuring other pluralities, which probably explains their widespread use. What we end up with are various concrete “numbers of things”, on the one hand, and “numbers” in the sense of numerals, on the other hand. Finally, these two senses of “numbers” and their relation were already well known in classical Greece, e.g., by Aristotle when he writes:

Number, we must note, is used in two senses – both of what is counted or countable and also of that with which we count. (Aristotle, 1984, 219<sup>b</sup>5-7)

“What is counted” are clearly pluralities or multitudes here, and “that with which we count” are numerals, I assume.



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Kant's and Mill's respective views about arithmetic are landmarks in the history of the subject, in the sense that most nineteenth- and even twentieth-century philosophers of mathematics develop their own ideas and positions in response to them. Frege, in particular, presents his conception of numbers in direct opposition to both. But before turning to Frege's views, I want to bring into play two additional developments in the nineteenth century. The first development is the gradual clarification of the notion of set, especially its separation from the notion of aggregate in something like Mill's sense; the second is the emergence of higher arithmetic, including the use of sets, even infinite sets, of numbers to construct other mathematical objects.

In connection with our earlier discussion of pluralities (multitudes, *arithmoi*), contemporary readers may find it tempting to translate "plurality" as "set", or more narrowly as "finite set". At the same time, some of our earlier observations may have given such readers pause, e.g., in connection with Mill's understanding of pluralities as concrete aggregates, agglomerations, or parcels, which seem rather different from sets in the current sense. But what exactly is the difference? It will help to reconsider Millian aggregates with respect to two distinctive features they have. First and as already mentioned above, for Mill such an aggregate, say a heap of ten rocks, is as concrete and physical as the rocks of which it consists. This means the following: the aggregate has a location in space and time; it can be affected causally; and it can be ascribed physical properties such as a weight, color, etc. Second and in addition, the relation of the aggregate and the individual rocks comprising it is best thought of as that between a whole and its parts. It is in that sense, then, that the heap consists of the rocks.

In both of these respects sets, as understood in contemporary set theory, are rather different. Sets are supposed to be abstract

objects, not concrete, physical ones; and that means that it doesn't even make sense to ascribe a spatio-temporal location, causal influence, or physical properties to them. A set also doesn't consist of its elements in the way in which a heap consists of the rocks in it. This latter difference becomes perhaps clearest if we observe several formally describable disanalogies between the part-whole relation and the element-set relation. Here are, briefly, four such disanalogies.

First, it is true of any whole that a part of a part of the whole is also a part of the whole; i.e., the part-whole relation is transitive. But the element-set relation is not transitive; an element of an element of a set is often not itself an element of the set. Second, with respect to sets there is an important and sharp distinction between the element relation and the subset relation; a subset of a set is in general not also an element of the set. In contrast, this distinction does not exist, or is not clear, in the case of the part-whole relation; several parts of a whole taken together can again be considered to be a part of the whole. Third, consider a singleton set, i.e., a set containing just one element, say a particular rock. According to our contemporary conception, the singleton set containing the rock is clearly different from the rock itself; since the singleton set has an element, while the rock doesn't have any elements (now apply the axiom of extensionality). On the other hand, it is again not clear what corresponds to a singleton set on the part-whole side; or while it is formally possible to consider a whole consisting of just one part, e.g., the same single rock, in that case the whole is identical with the part, as both just consist of the rock. Fourth and finally, it is even harder to come up with something corresponding to the empty set on the part-whole side. In fact, a tendency to confuse sets with aggregates in the sense of wholes is what often leads to hesitation or suspicion about

the empty set, while from a clearly understood set-theoretic perspective the seeming incoherence vanishes.<sup>17</sup>

Given such basic differences between the two notions – aggregate versus set – a question arises with respect to our earlier discussion. Namely, in which sense should we understand the notion of plurality (multitude, *arithmos*) as employed by thinkers before Mill, and especially before the nineteenth century: as a (finite) set in the contemporary sense, or as an aggregate in the part-whole sense? Because this involves a lot of different thinkers, a careful, detailed answer will probably be complicated, and will vary from case to case. But I assume that the following is basically correct in general: It is typically not clear how to understand the notion of plurality as employed before the nineteenth century, especially in the respect at issue. The way in which it is invoked tends to be ambiguous, and not infrequently aspects of both relevant notions are mixed together. It simply took a long time for the contemporary notion of set to be separated, clearly and sharply, from that of aggregate.<sup>18</sup> Crucial clarifications in this connection were due to Bolzano, Cantor,

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<sup>17</sup> As a fifth and more general difference, one could also mention the following: Unlike in the case of set theory, where there is essentially only one standard way of systematizing and axiomatizing the theory (at least its basic parts, i.e., putting aside large cardinal questions and other more advanced issues), the part-whole relation allows for several significantly different systematizations none of which is clearly privileged (even considering only basic aspects). At the same time, those aspects of the part-whole relation appealed to above in distinguishing it from the element-set relation are common to all systematizations, as far as I am aware. For more on the systematic study of parts and wholes, i.e. mereology, see Simons, 1987.

<sup>18</sup> As late as 1903, in his *Principles of Mathematics*, Bertrand Russell still struggles to make the distinction clear; compare chapters 6, 16, and 17 in Russell, 1903.

Dedekind, and Frege, among others, all in the nineteenth century. But even in the writings of seminal thinkers such as Bolzano, Cantor, and Dedekind, one can find the identification of a singleton “set” with its element, as well as hesitations about the empty “set”, thus again muddying the waters. Also, sets, best thought of as abstract objects, are discussed by them as if they were mental objects, which also muddies the waters significantly.<sup>19</sup>

Nevertheless, by the end of the nineteenth century the element-set relation had been studied and elaborated in considerable detail, and basically distinguished from the part-whole relation. A large part of the motivation for these studies was the new use of sets in clarifying various mathematical notions, such as those of the negative, rational, and real numbers (Cantor, Dedekind, etc.), and for introducing notions such as that of an ideal in algebra (Dedekind). Some of the corresponding constructions, e.g., those for the real numbers and for ideals, not only involve finite sets, but even, crucially and unavoidably, infinite sets. Note that this leads beyond the notion of aggregate in, say, Mill’s sense; since it would surely stretch that notion beyond what he had in mind to consider “infinite aggregates”, given his very concrete way of thinking about them.<sup>20</sup>

A particularly important aspect of such uses of sets for our purposes is the following: In the constructions of the real numbers, say, we use infinite sets of rational numbers; and that means that the rational numbers are treated as individual objects, because that is what the elements of such sets are presumed to be. Similarly, in the

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<sup>19</sup> An interesting case is Dedekind; compare Reck, 2003 in which I argue against a corresponding psychologistic reading of his writings, despite superficial evidence to the contrary.

<sup>20</sup> Having said that, it should be acknowledged that it is possible, as was made clear relatively recently, to develop a theory of infinite mereological wholes or sums; compare Lewis, 1991.

construction of the rational numbers out of the integers we treat the integers as individual objects; and in the construction of the integers out of the natural numbers we treat the natural numbers as individual objects. Now, how could one reconcile this treatment of numbers, including the natural numbers, with our earlier conceptions? Should they be identified with numerals in this connection, since numerals are, or can be treated as, individual objects? Well, doing so puts significant pressure on the idea of a numeral, at least if it is understood in a concrete way, since we need an actual infinity of them. But if we don't want to identify numbers with numerals, what alternative remains?

The use of infinite sets of numbers as just described is part of higher arithmetic, the extension of elementary arithmetic that leads over to the study of the negative, rational, and real numbers. There are other aspects of higher arithmetic that lead to the treatment of the natural numbers as individual objects as well, although perhaps not in as clear and compelling a way. The general, underlying issue here is this: When we merely consider elementary arithmetic, especially simple numerical equations such as " $2 + 3 = 5$ ", it seems natural, or is at least possible, to analyze them semantically in a way that does not involve treating numbers as objects. Again, we can treat "2", "3", etc. as "common names" for corresponding pluralities. Now, we may be able to extend this kind of semantic analysis systematically to more complex arithmetic sentences such as: "for all  $n$  and  $m$ ,  $n + m = m + n$ " or "for all  $n$  there is a  $p$  such that  $p > n$  and  $p$  is prime". But already in these cases, we are dealing with complicated nested generalities as a result, explicitly or implicitly. And if we also allow for higher-order quantification in arithmetic sentences (as in the case of the full axiom of induction), it is not clear any more how to extend the semantics. Put briefly, the further we go beyond elementary arithmetic, the less an analysis in terms of common names will seem natural and appropriate, or even possible.

Instead, it will suggest itself more and more to understand terms such as "2", "3", etc. as singular terms, thus as referring to particular objects. In fact, even within elementary arithmetic a statement such as "5 is prime" seems naturally analyzed as having the form  $P(a)$ , i.e., as involving the attribution of a property (to be prime) to an object (the number five). Then the question arises, once more, what the object involved is: the numeral "5" or something else instead?

Actually, at this point another suggestion may be made. Why not combine our clarified notion of set with the idea, mentioned earlier, of using Platonic "pure units"? If we do that, the following becomes an option: We can treat the number five as the set containing five such units; similarly for all the other natural numbers. We can even treat infinite numbers along the same lines, at least if we accept the existence of infinitely many pure units. Now, this is exactly the conception of numbers one can find, at the end of the nineteenth century, in the works of Georg Cantor.<sup>21</sup> According to it, numbers do turn out to be individual objects, but they are not identified with numerals. Note also that with this conception of numbers we have moved quite far away from our naïve, practical views about arithmetic. Even with a simple arithmetic statement such as " $2 + 3 = 5$ " we are now no longer talking about concrete objects in the physical world, but about abstract sets of pure units. Finally, this conception of numbers will only be attractive for someone who finds the notion of a pure, abstract unit unproblematic.

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We are finally in a position to reconsider Frege's conception of the natural numbers, as first presented in *Die Grundlagen der*

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<sup>21</sup> See the discussion of Cantor's conception of numbers in Tait, 1997; compare also Fine, 1998.

*Arithmetik* (Frege, 1884) and spelled out further, as well as modified slightly, in *Grundgesetze der Arithmetik* (Frege, 1893/1903). Actually, let me start by introducing what is often called the “Frege-Russell conception”, before then turning to Frege’s particular version, or versions, of it.

Given the ideas and developments discussed so far, the “Frege-Russell conception” can be seen as a fairly natural and direct extension of earlier views. Here is the basic idea: We started out with the conception of numbers as pluralities of things. The numeral “2”, say, corresponds then to all the two-numbered pluralities, in the sense that it is used as a “common name” for them. Disambiguating the notion of plurality, or better, replacing it by the clearer notion of set, we can modify this view by saying that the numeral “2”, as used in arithmetic statements, corresponds to all two-element sets. Now we want to push the modification one step further, for reasons having to do with higher arithmetic. Namely, we want to treat the numeral “2” as referring to a particular object. What object could that be (putting aside sets of pure units, as well as the identification of numbers with numerals)? Well, this new object should be related to all the two-element sets in some intimate, uniform way. Why, then, not simply collect together all those sets, i.e., why not form the set of all two-element sets? This is an infinite set, to be sure. But we have been using infinite sets in higher arithmetic already anyway, so that it appears to be a legitimate object. Moreover, it is closely related to all two-element sets in a uniform way, by containing them as elements. Finally, we can use the same procedure for all the other natural numbers as well, indeed even beyond them.

A classic presentation of this conception of the natural numbers can be found in Bertrand Russell’s *Introduction to Mathematical Philosophy* (Russell, 1919). Using the terminology of “class”, “collection”, and “bundle”, instead of “set”, he writes:

[N]umber is a way of bringing together certain collections, namely, those that have a given number of terms. We can suppose all couples in one bundle, all trios in another, and so on. In this way we obtain various bundles of collections, each bundle consisting of all the collections that have a certain number of terms. (Russell, 1919, p. 14)

It is interesting, after our earlier aside, that Russell starts with the number 2 here, not with 0 or 1. A few pages later, however, he makes clear that the same approach can be used in the case of those two numbers as well:

We [also] want to make one bundle containing the class that has no members: this will be for the number 0. Then we want a bundle of all the classes that have one member: this will be for the number 1. (*ibid.*, p. 17)

In Russell’s writings, this conception is presented for the first time in “The Logic of Relations” (Russell, 1901), then also in *Principles of Mathematics* (Russell, 1903, p. 115ff.). However, Russell was neither the only nor the first thinker to introduce it. Even putting aside Frege’s works for the moment, the same basic idea occurs in the works of other thinkers as well, and as early as the 1880s. For instance, in an 1888 letter to Richard Dedekind (published posthumously) the mathematician Heinrich Weber proposes, independently of Frege and Russell, the use of equivalence-classes of classes to define the natural numbers.<sup>22</sup> Such a view was simply in the air at the time, it seems.<sup>23</sup>

<sup>22</sup> Dedekind’s response to this letter, which contains a brief description of Weber’s proposal, is published in Dedekind, 1932, p. 488-90; compare the discussion in Reck, 2003, section 6.

<sup>23</sup> Mark Wilson has connected Frege’s use of equivalence-classes of classes for defining numbers to related constructions in nineteenth-century geometry, especially projective geometry; see Wilson, 1992. I am inclined

Nevertheless, this conception of numbers is most well known from Russell's writings, and it is usually also attributed to Frege; thus the name "Frege-Russell conception". Frege did, indeed, hold corresponding views, at least in *Grundgesetze der Arithmetik*. I now want to reexamine certain aspects of these views, including his criticisms of other conceptions of numbers.

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The text in which Frege discusses rival conceptions of the natural numbers most explicitly is *Die Grundlagen der Arithmetik* (Frege, 1884). His discussion of them is almost exclusively critical and polemical. Among the main targets of Frege's criticisms are: simple formalist views; Mill's empiricist and physicalist position; and the conception of numbers as sets of pure units.<sup>24</sup> Without going into the details of all of his arguments, let me say enough about them to establish the connection to our earlier discussion. Among Frege's main objections to simple formalist views, especially views that identify numbers with numerals, is the following: It is not the numerals *qua* physical objects that are of interest to us in arithmetic; for example, their physical properties (shape, color, etc.) do not matter as far as arithmetic truth is concerned. Rather, what matters is our specific use of the numerals, or better of the corresponding numbers, both in arithmetic itself and in its applications.

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to believe that Frege was influenced along those lines. But if the account proposed in the present paper is correct, he needn't have started from geometry to be led in this direction.

<sup>24</sup> See chapters I-III of Frege, 1884. I leave aside Frege's criticisms of psychologistic views, since they are less relevant for present purposes; compare Reck, 1997 for a corresponding discussion.

More particularly, what matters is their use in measuring "numbers of things", i.e., in determining the size of pluralities.

With respect to earlier treatment of such pluralities, Frege formulates two basic, related criticisms. Consider Mill and his appeal to concrete, physical "aggregates", as well as to corresponding operations of "separation and rearrangement". Such appeals are to be rejected, according to Frege, both because they misrepresent the wide range of applicability of arithmetic and because they make arithmetic dependent on empirical considerations in a problematic way. As to the former, note that numbers cannot merely be applied to physical aggregates of things; we can also number, say, historical events, mental images, and even non-spatio-temporal objects such as abstract proofs of a theorem. As to the latter, the truth of an arithmetic statement such as " $2 + 3 = 5$ " does not depend on our ability, much less on the actual performance, of physical operations. According to Frege, what needs to be done to avoid both kinds of mistakes is to replace the notions of aggregate and arrangement, as understood by Mill and others, by corresponding logical notions.

This is, of course, exactly what we did above by introducing sets and set-theoretic operations (union, intersection, etc.). Frege does not appeal to sets, however, at least not directly. There are two main reasons why he does not do so. First, in his critical survey of earlier views he does not find any clear notion of set; all he finds are ambiguity and confusion. A particular symptom of the confusion is the way in which the numbers 0 and 1 are treated, or rather excluded from treatment. Frege's general assessment in this connection is the following:

Some writers define the number as a set or multitude or plurality. All of these views suffer from the drawback that the concept will not then cover the numbers 0 and 1. Moreover, these terms are

utterly vague: sometimes they approximate in meaning to "heap" or "group" or "aggregate", referring to a juxtaposition in space, sometimes they are so used as to be practically equivalent to "number", only vager. (Frege, 1884, p. 38)

As a result of such "utter vagueness", ambiguity, and related problems, the very term "set" is suspicious to him.<sup>25</sup> From the 1890s on, Frege himself starts to appeal to the notion of "class". However, he does so only in terms of a reconstructed and derivative version of that notion. This leads us to his second main concern. Namely, for Frege the fundamental notion is not that of a set or class, even in its clarified form, but the notion of a concept. A set or class – in his preferred terminology, an "extension" – is always something determined by a concept; it is always the "extension of a concept". In other words, Frege works with what we now call the "logical" notion of set or class.

One immediate benefit of making concepts central, as opposed to relying on the earlier, ambiguous notion of plurality or multitude, is that Frege can provide a significant clarification of the notion of (concrete) unit. We noted above that, strictly speaking, a plurality or multitude can only be numbered – can only be considered a "number of things" – relative to a "partition" or "choice of unit". It was left unclear, however, what the latter involves. Frege proposes the following analysis: In numbering, we rely on a "choice of unit" in the sense that we consider items as

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<sup>25</sup> In Tait, 1997 it has been argued (against Dummett) that Frege's discussion of earlier views is often uncharitable and tends to be unfair. I think this is true to some degree in the present connection, but it is also true that a large number of earlier writers really do exhibit confusion, or at least vagueness and ambiguity, in their use of "multitude", "plurality", "aggregate", "collection", or "set".

falling under a certain concept.<sup>26</sup> For example, in a pack of cards each card can be counted as one, and be seen as equal to every other card in that respect, insofar as it falls under the concept "card in the pack". This is what underlies assigning a corresponding number to the pack of cards. Note that, if we switch to the concept "complete deck of cards in the pack", what counts as one changes, thus the number to be assigned changes, even if we are still looking at the same pack of cards. For Frege, this observation leads to an additional criticism of positions such as Mill's that rely on mere groups (aggregates, parcels, heaps, etc.) of things as the basis for the concept of number. But it also leads to the more positive claim, central to Frege's logicism, that a statement of number is always a statement about a concept. A second immediate benefit of Frege's focus on concepts is this: It is not hard to come up with concepts under which exactly one object falls; similarly for concepts under which no object at all falls. As a result, Frege has no problems acknowledging 0 and 1 as numbers, on a par with 2, 3, 4, etc.

Interestingly, Frege not only considers the notion of a multitude of concrete units in his criticisms of earlier views, but also that of a multitude of abstract units. As noted above, the corresponding conception of numbers was introduced already in Plato's dialogues, and it was reintroduced, in an updated form, in Cantor's writings from the late nineteenth century. With respect to

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<sup>26</sup> Actually, the term "unit" is ambiguous here. On the one hand, there is the "unit of measurement", i.e., the thing that provides the relevant individuation, identified by Frege as the concept. On the other hand, there are the individual items to be numbered, or to be counted as "units" in the sense of "ones". Frege's emphasis is usually on the concept as constituting the "unit" in the first sense, since this is the ingredient missing in views such as Mill's; cf. Frege, 1884, §54. (I owe this clarification to Marco Ruffino.)

the pre-Cantorian version of this view, a first criticism raised in Frege's *Grundlagen* involves, once more, the problem that the notion of multitude needs to be clarified and logicized. But even if we move over to Cantor's version, as based on the notion of set, a second problem remains from Frege's point of view. Namely, it is not clear how to make sense of the idea of several numerically distinct, but otherwise indistinguishable units. In Frege's own words:

If we try to produce the number by putting together different distinct objects, the result is an agglomeration in which the objects contained remain still in possession of precisely those properties which serve to distinguish them from one another; and that is not the number. But if we try to do it in the other way, by putting together identicals, the result runs perpetually together into one and we never reach a plurality. (*Ibid.*, p. 50)

In other words, if the units are really and totally indistinguishable, as the view at issue has it, they all coincide, don't they (by Leibniz's law of the identity of indiscernibles)? But then there seems to be only one unit left, so that the idea of a plurality evaporates.

My goal in this essay is not to assess the stringency of Frege's criticisms, but to illuminate the relation of his conception of numbers to other such conceptions, especially earlier ones.<sup>27</sup> What we have found so far, in connection with the criticisms, is that Frege replaces the notion of plurality or multitude not by the notion of set, but by his alternative notion of concept and, derivatively, by the notion of extension of a concept. This leads him to the thesis that a statement of number amounts to a statement about a concept. Two further steps have to be added now to arrive at Frege's own

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<sup>27</sup> For a critical discussion of Frege's arguments against, among others, the Cantorian conception of numbers, see again Tait, 1997. For another recent defense of Cantor's position, see Fine, 1998.

conception of numbers: first, his appeal to certain logical relations between such concepts, in particular the relation of equinumerosity; second, his move from equinumerous concepts to corresponding logical objects.

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As noted earlier, the notion of equinumerosity was already invoked long before Frege; or better, a concrete, non-logicized version of this notion was used, in connection with the corresponding concrete notions of plurality or multitude of units. In *Die Grundlagen der Arithmetik*, Frege himself points to this earlier usage when he refers to Hume's definition of numerical equality:

Hume long ago mentioned: "When two numbers are so combined as that the one has always a unit answering to every unit of the other, we pronounce them equal." (Frege, 1884, p. 73)<sup>28</sup>

This passage is the source of what has come to be called "Hume's Principle" in recent neo-Fregean investigations. From the point of view developed in the present paper, however, the following two limitations of Hume's proposal should be clear now: First, it is based on the notion of a unit, or of a multitude of units, as understood in the ambiguous pre-nineteenth-century sense. Second, the relation referred to by the phrase "a unit answering every unit of the other" is not analyzed further, especially not in Fregean logicist or related set-theoretic terms. There is then a question of how to conceive of this relation: in concrete Millian terms or in

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<sup>28</sup> See Hume, 1988, Bk. I, Part iii. Section 1, for the original source.



some more abstract way?<sup>29</sup> Such ambiguities are completely resolved in Frege's works. In particular, equinumerosity now amounts to the existence, in his logical system, of a bijective function between two concepts. Similarly, an ordering relation between concepts in terms of their cardinality can be introduced, in terms of the existence of corresponding injective functions, etc.

We have still not made the step, along Fregean lines, to conceive of numbers as particular objects. Actually, there is another step, also indicated by Frege, that leads in a somewhat different direction and is relevant as well. Namely, we can introduce "numerical concepts", in the sense of second-order concepts usable for assessing the size of first-order concepts (at least in the finite case). For instance, there is the concept of "two-ness", definable logically thus:  $\exists x \exists y (Fx \ \& \ Fy \ \& \ x \neq y \ \& \ \forall z (Fz \rightarrow (x = z \vee y = z)))$ ; similarly for "three-ness", "four-ness", etc., and even for "one-ness" and "zero-ness". These notions have the following immediate connection to equinumerosity: two (finite) first-order concepts fall under the same second-order numerical concept if and only if they are equinumerous. What that means is that for each equivalence class of (finite and) equinumerous concepts we get a corresponding numerical concept. Note, furthermore, that the numerical concepts of "two-ness", "three-ness", etc. corresponds relatively closely to the uses of "two", "three", etc. as common names, in the sense discussed above. In other words, Frege's system has the resources to provide us with updated, logicized analogues of such common names.

At this point, it may be tempting to identify the natural numbers with numerical concepts, especially if one has logicist

<sup>29</sup> A third, related limitation of Hume's position is that he, unlike Frege and especially Cantor, only considers finite "numbers"; cf. again Tait, 1997 in this connection.

leanings.<sup>30</sup> This is not what Frege does, however, in spite of having the latter at his disposal. The problem is that these are concepts, not objects; and he insists that numbers are, or have to be identified as, objects. The explicit reasons Frege gives for this insistence, especially in *Die Grundlagen der Arithmetik*, are grammatical. He argues, in particular, that the primary use of numerals and number words, in ordinary and in mathematical discourse, is as object names, not as concept names. Beyond that, our earlier discussion has revealed another (not unrelated) motivation he may have had as well: in higher arithmetic numbers are treated as objects in certain crucial constructions. Finally, in Frege's own reconstruction, as spelled out further in *Grundgesetze der Arithmetik*, it is important to treat numbers as objects at certain crucial junctures, e.g., in the proof that each number has a unique successor.

Whatever the main motivation for this step, Frege goes on to introduce numbers as equivalence classes with respect to the equivalence relation of equinumerosity, both in *Grundlagen* and in *Grundgesetze*. As such, numbers are explicitly identified as a certain kind of logical objects. Actually, the relevant definitions in these two works are not identical, as one may assume at first. In *Grundgesetze*, numbers are introduced as equivalence classes of concepts. More specifically, the number 0 is defined as the equivalence class of all concepts equinumerous to the concept  $x \neq x$  (a concept under which no object falls); the number 1 is defined as the equivalence class of all concepts equinumerous to the concept  $x = 0$  (a concept under which exactly one object falls); the number 2 is defined as the equivalence class of all concepts equinumerous to the concept  $x = 0 \vee x = 1$ , etc.<sup>31</sup>

<sup>30</sup> Compare the corresponding discussion of "numerical quantifiers" in Bostock, 1974 and Hodes, 1984.

<sup>31</sup> Officially Frege defines 0 as indicated, but then 1 as the successor of 0, 2 as the successor of 1, etc. (after introducing a corresponding successor



Note that 0 is then exactly the extension of the second-order concept “zero-ness”, 1 the extension of “one-ness”, 2 the extension of “two-ness”, etc. More generally, the number belonging to a concept  $F$  is the extension of the concept “equinumerous to  $F$ ”.

What we are presented with in *Grundlagen* is, thus, not exactly the “Frege-Russell conception of numbers”; since what is contained in the relevant equivalence classes are concepts, not sets or classes. It is only in *Grundgesetze* that Frege moves over to the use of equivalence classes of classes.<sup>32</sup> And even then, it remains crucial for him to think of the corresponding classes as extensions of concepts; likewise for his continued treatment of equinumerosity as a relation between concepts, not between extensions, etc. To be more explicit about the change in *Grundgesetze*: The number 0 is now defined as the equivalence class of all classes whose corresponding concepts are equinumerous to  $x \neq x$ ; thus 0 is the extension of a first-level concept corresponding to “zero-ness”, but not the extension of “zero-ness” itself. Similarly for 1, 2, 3, etc. More generally again, the number belonging to the concept  $F$  is now the extension of the concept under which fall those extensions of concepts equinumerous with  $F$ .<sup>33</sup>

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function); see part IV of *Grundlagen*, especially §§74-77. The overall result is the same, though.

<sup>32</sup> I think it is an interesting question why Frege made this change. Probably part of the motivation is technical: It simplifies the logical system in *Grundgesetze*; in particular, the notion of extension has to be introduced only for first-level concepts, not also for higher-level concepts. (Here, and in related remarks below, I am indebted to Kai Wehmeier.) In addition, compare the discussion of the central role played by extensions of concepts in Frege’s logicism, as motivated by philosophical reasons, in Ruffino, 2003.

<sup>33</sup> See *Grundgesetze der Arithmetik*, §40 etc. This account of the relation between the conceptions of numbers in Frege’s *Grundlagen* and *Grund-*

My main goal so far has been to establish connections between Frege’s logicist conception, or conceptions, of numbers and the earlier non-logicist notions of plurality, equinumerosity, etc. I have tried to show, more particularly, how Frege’s conception can be seen to grow naturally and relatively directly out of earlier ideas and developments. At the same time, I do not mean to deny that significant changes have taken place in Frege’s work. Besides the replacement of the notion of plurality or multitude by that of concept or class and besides the related logicist modifications of the notions of equinumerosity etc., it is especially the use of (infinite) equivalence classes in connection with the natural numbers that constitutes a crucial change. It should also be emphasized that, while Frege was not the only mathematician at the time to make use of the technique of forming equivalence classes, he was one of the first, perhaps even the first, to be aware that this technique stands in need of further analysis and defense. In fact, one can see the development of crucial parts of Frege’s new logic, especially of the theory of extensions contained in it, as an attempt to provide a systematic foundation for it. Finally, Russell’s antinomy confirms that something substantive, and problematic, is going on here.

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Frege’s conception of the natural numbers is connected to the earlier notion of plurality or multitude especially in the following respect: logicized variants of such pluralities, namely corresponding concepts or classes, are built right into his main definitions. I emphasized above that, if one conceives of numbers as pluralities, especially as concrete pluralities, the umbilical cord between

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*gesetze* agrees with, and is partly influenced by, that in Blanchette, 1994; compare especially her footnotes 29 and 49.

arithmetic and physical reality is firmly in place. What should we say about Frege's conception in this connection? In one respect his replacement of pluralities with concepts or classes does affect, and perhaps loosen, the umbilical cord. Namely, for him a numerical statement is no longer directly about physical things, but about abstract concepts. However, insofar as we apply numbers to the world via these concepts, the application of arithmetic is still built into Frege's definitions. Perhaps we should say, then, that the original umbilical cord has been transformed into, or replaced by, a more abstract bond. Crucially, there still is a bond – arithmetic and its applications have not been cut apart completely. In other words, Frege's conception does not go all the distance towards "pure arithmetic", compared to the way in which Hilbert's conception of geometry goes all the distance towards "pure geometry".

Looked at from another angle, what is at issue here is the following: Frege is well aware that his definitions and constructions provide, or are meant to provide, a particular natural number sequence, i.e., a particular "simple infinity" (Dedekind), "progression" (Russell), or " $\omega$ -sequence" (Zermelo). In modern terminology, they specify a particular model of the Dedekind-Peano Axioms. Now, for purely inner-mathematical purposes all that matters about such models, or about the elements in them, is their relational or structural properties, those that can be derived simply from the axioms; any further, intrinsic properties such elements may have, e.g., that each number except for 0 contains infinitely many concepts or classes, are irrelevant. On the other hand, it is exactly by means of these latter properties that the application of arithmetic is built into Frege's conception. As this shows, Frege's natural numbers combine two different, separable aspects or ingredients within themselves. One ingredient concerns what is needed for pure arithmetic, the other what is needed for its applications.

The step to separate these two ingredients, completely and cleanly, was first taken by a contemporary of Frege's: Richard Dedekind. In Dedekind's classic essay "*Was sind und was sollen die Zahlen?*" (Dedekind, 1888), the goal is to distill out what is crucial for inner-mathematical purposes alone, i.e. for pure arithmetic, and to leave everything else aside. To do so Dedekind first defines, within an informal background theory of sets and functions, the (higher-order) concept of a "simple infinity". Then he constructs, parallel to Frege, a particular simple infinity, or a particular natural number sequence. In a third step, he "abstracts away" from the intrinsic nature of the elements in that sequence. The result of this three-step procedure is an abstract structure, specifically that of the natural numbers. Dedekind conceives of this structure as a distinctive natural number sequence, one whose elements no longer have properties that are "foreign" to pure arithmetic, as Frege's equivalence classes do.<sup>34</sup> It is only at a later, secondary stage that Dedekind adds an account of how one can recover the application of the natural numbers as cardinal numbers within his system. He does so by showing how to use initial segments of his number sequence as "tallies", where the relevant tallying is to be understood in set-theoretic terms.

As just described, Dedekind's conception of the natural numbers is a structuralist conception. It is, however, not the only possible structuralist position. Indeed, it is not the most well-known and widespread one today. A more common alternative is what has been called "set-theoretic structuralism". Here one starts by constructing, within Zermelo-Fraenkel set theory, the finite von Neumann ordinals  $\omega$ , starting with  $\emptyset$  and continuing with the successor function  $f: x \rightarrow x \cup \{x\}$ ; and then one treats these sets as

<sup>34</sup> Such an interpretation of Dedekind's position in Dedekind, 1888 is not uncontroversial, in general and with respect to this particular aspect. A detailed defense of it is presented in Reck, 2003.

“the natural numbers”, at least pragmatically. This becomes a structuralist position if two further claims are added: first, that there are various other set-theoretic natural number sequences one could use instead, e.g., the finite Zermelo ordinals (based on the alternative successor function  $g: x \rightarrow \{x\}$ ); second, that all that matters, in the end, is that the sequence used forms a model of the Dedekind-Peano Axioms.<sup>35</sup> A related, but more radical alternative, also quite widespread these days, is to insist that what really matters is simply the axioms and what follows from them, where the latter is to be understood in some sophisticated formalist or “if-then-ist” sense.<sup>36</sup>

For present purposes, the crucial point is the following: Along all such structuralist and sophisticated formalist lines, whatever the further details, we have separated pure arithmetic completely from its application to physical reality. The result can be described by paraphrasing Freudenthal’s remark about geometry:

The umbilical cord between reality and arithmetic has been cut. Arithmetic has become pure mathematics, and the question whether and how it can be applied to reality is answered just as in any other branch of mathematics. The axioms are no longer evident truths, indeed it doesn't even make sense any more to ask about their truth.

It seems fair to say that this general result is accepted widely among mathematicians today, explicitly or implicitly, especially those working within a set-theoretic framework. One consequence is, as in the case of geometry, that it brings arithmetic in line with other parts of pure mathematics. More particularly, arithmetic, like

<sup>35</sup> See Reck & Price, 2000 for a discussion of this variant of structuralism (among others).

<sup>36</sup> See Rheinwald, 1984 for a systematic discussion of corresponding positions.

geometry earlier, becomes part of the general study of structures or relational systems.

Frege, to emphasize it again, does not go that far. For him arithmetic is not just concerned with an abstract structure or with purely formal facts, neither in the sense of Dedekind or other structuralists, nor along sophisticated formalist lines. Rather, the subject matter of arithmetic is a system of particular abstract objects, objects that have it built into their very nature how arithmetic is applied. Moreover, from this point of view the attempt to separate pure arithmetic completely from its applications, even if possible, is not a virtue but a vice – it makes us miss something about the concept of number, an aspect that is distinctive about it. I do not want to decide here whether such a Fregean claim can be defended or not. But one thing should be clear by now: any of the structuralist or formalist positions just mentioned is much farther away from our naïve, practical conception of numbers than Frege’s position is.

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Frege’s attempt to keep arithmetic and reality together – to preserve the umbilical cord between them, even if in a more abstract or logical form than before – is certainly ingenious, in a number of ways. As I have tried to make evident, it is a subtle attempt to combine what is right about an old, naïve conception of numbers with new, nineteenth century clarifications and transformations in arithmetic, logic, and set theory. Alas, it is also inconsistent, as Russell’s antinomy shows. Most discussions of Frege’s conception subsequent to Russell’s discovery have focused on questions about consistency and inconsistency. My focus has been different. I have tried to situate Frege’s conception between earlier practical conceptions and later pure ones. One benefit of doing so is that its motivation, its core ideas, and their significance appear in a new

light. More negatively, Frege's conception of the natural numbers has revealed itself as not entirely in line with much of contemporary mathematics, at least insofar as the latter tends, in practice, to be informed by structuralist or formalist views.

At this point it may seem that Frege's conception of the natural numbers is, in the end, mostly a failure: not only is it inconsistent, it also doesn't quite break through to a pure, completely structural or formal, perspective on arithmetic. Why, then, should we still concern ourselves with it? I think it is worth doing so, for at least three reasons. The first reason has to do with a central theme of this paper. Frege's conception constitutes an interesting link between the earlier conception of numbers as pluralities or multitudes and currently prevalent structuralist and formalist conceptions. Note here that, insofar as the pluralities conception still shapes how a lot of people, especially people who are not mathematicians or philosophers of mathematics, think about numbers, current structuralist or formalist conceptions of numbers can seem unmotivated and strange. If we bring in Frege's conception for comparison, or perhaps for a kind of triangulation, it allows us to understand their relationship better.

Second, Frege's position is, as we have seen, built around a subtle, detailed analysis of the application of arithmetic, especially of the use of numbers as cardinal numbers. In contrast, from contemporary structuralist or formalist points of view the application of arithmetic is separated from arithmetic itself – and then the focus is almost exclusively on the latter. This new, exclusive focus on pure arithmetic has, no doubt, led to many results that are important, both mathematically and philosophically, such as Gödel's Theorems. But it has the disadvantage that potentially also interesting questions about the application of arithmetic tend to be ignored, or swept under the carpet. This leads to the question: Is it really the case that there are no important philosophical problems

on the application side?<sup>37</sup> More basically, how exactly are we to think about the application of arithmetic along structuralist or formalist lines?<sup>38</sup> A reflection back on Frege's conception of numbers may urge such questions on us.

A third and final reason for still concerning ourselves with Frege's conception of numbers has to do with current debates in the philosophy of mathematics. Recently there have been serious attempts to modify and revive Frege's conception, in such a way as to avoid Russell's antinomy but preserve many of its seemingly attractive features. I am referring to the neo-Fregean approaches based on "abstraction principles", especially "Hume's Principle", developed in the works of Crispin Wright, Bob Hale, George Boolos, Richard Heck, and others. Crucially, these approaches share with Frege's the feature of trying to preserve the umbilical cord

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<sup>37</sup> In the recent literature, see Steiner, 1998 for an extended argument that we should reconsider the application of mathematics philosophically; see also Tetens, 1994 for a contemporary exploration of related Kantian ideas. In Wittgenstein, 1978 similar issues are addressed, among others, as both Steiner and Tetens acknowledge. (I owe my emphasis on issues concerning application to Mike Price.)

<sup>38</sup> Both in Dedekind's and in set-theoretic structuralist texts there are usually hints at how to think about applications. But typically they are not expounded further, in particular not with respect to philosophical aspects. Compare in this connection my own hint at Dedekind's technique of using initial segments of his natural numbers as "tallies". Doing so is, for him, to be reconstructed within an informal theory of sets and functions. An interesting question might be, then, how this account, if spelled out in more detail, compares to Frege's views about applications. Note that Dedekind's is not exactly the contemporary set-theoretic position, among others because he does not reduce functions to sets. Indeed, for Dedekind, as for Frege, to think in terms of functions is basic for human thought; see the Preface to Dedekind, 1888.

between arithmetic and reality.<sup>39</sup> From the point of view developed in the present paper, one then wants to ask not only how exactly such neo-Fregean approaches compare to Frege's original one, but also how they relate to earlier practical and later structuralist and formalist conceptions. Both seem to me to be interesting questions that have not found much attention in the literature yet. But after having raised them, I will have to leave exploring these questions for another occasion.

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<sup>39</sup> The fact that applications are built right into the corresponding conceptions of numbers, via Hume's Principle etc., is often presented as a philosophical advantage by their proponents; see, e.g., the introduction to Hale & Wright, 2001. Compare also the corresponding papers in Demopoulos, 1995.

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