FREGE

Frege on Numbers: Beyond the Platonist Picture

By Erich H. Reck

GOTTLOB FREGE IS OFTEN CALLED A "PLATONIST." IN CONNECTION WITH HIS philosophy we can talk about Platonism concerning three kinds of entities: numbers, or logical objects more generally; concepts, or functions more generally; and thoughts, or senses more generally. I will only be concerned about the first of these three kinds here, in particular about the natural numbers. I will also focus mostly on Frege's corresponding remarks in *The Foundations of Arithmetic* (1884), supplemented by a few asides on *Basic Laws of Arithmetic* (1893/1903) and "Thoughts" (1918). My goal is to clarify in which sense the Frege of *Foundations* and *Basic Laws* is a Platonist concerning the natural numbers.[1]

My strategy will be to look at Frege's Platonism "in context." To do so seems to me important because a direct, naïve approach to Platonism often leads nowhere, or at least not very far. Furthermore, Frege's corresponding views are not naïve, as I will try to show. (What is meant by "naïve" here will become clear shortly.) For that purpose I will contextualize Frege's Platonist statements in the sense of considering them in connection with his general approach in *Foundations* and *Basic Laws*. Connected with that I will distinguish two very different ways in which Platonism in itself can be understood. The "context" I have in mind thus consists in Frege's general approach, supplemented by a differentiated understanding of Platonism.

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IN THE *ENCYCLOPEDIA OF PHILOSOPHY*, VOLUME 5, STEPHEN BARKER GIVES THE FOLLOWING short and suggestive characterization of Platonism in mathematics:

> By platonism is understood the realistic view, akin to that of Plato himself, that abstract entities exist in their own right, independently of human thinking. According to this view number theory is to be regarded as the description of a

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realm of objective, self-subsistent mathematical objects that are timeless, non-spatial, and non-mental. Platonism conceives it to be the task of the mathematician to explore this and other realms of being. Among modern philosophers of mathematics Frege is a pre-eminent representative of platonism, distinguished by his penetrating lucidity and his intransigence. (p. 529)[2]

Note how the natural numbers are here described as "abstract objects" that are "self-subsistent," in the sense of existing "independently of human thinking"; how a separate "Platonic world" inhabited by such objects is conjured up; and how the mathematician is depicted as a "discoverer" or "explorer" of this world. These ingredients, or phrases, are typical for many other, often even shorter, characterizations of Platonism.

Usually it is then assumed that such short characterizations describe a definite philosophical position: Platonism. But what they really give us, in my view, is just a vague, even if suggestive picture (what I have called elsewhere, following Wittgenstein, a "metaphysical picture"): the picture of a "Platonic world"—or of a "Platonic heaven"—parallel to the world of physical objects. Based on this picture, Platonism is then often criticized and not seldom quickly dismissed. The main point of criticism is that this view leads to an "access problem"; that is, it is not clear how we can ever gain access to such a Platonic world.[3] However, for me a different, more basic question arises: Is this picture really definite enough for a criticism of Platonism or also for a corresponding defense? I do not think so. Rather, it seems to me to be too general, vague, and ambiguous (more on this issue later).

At the same time, it is hard to deny that Frege is a Platonist in this general, vague sense. To be sure, he never uses the term "Platonism" himself, and he rejects the characterization of the natural numbers as "abstract objects" since he takes the underlying notion of "abstraction" to be problematic. But, at least from *Foundations* on, he defends the following Platonist theses: numbers are independent "logical objects"; as such they are different from numerals and other physical objects, on the one hand, and from mental objects and psychological processes, on the other hand; furthermore, arithmetic is a science in which we refer to such objects with our number words and numerals, in which we ascribe properties to them, and in which we thus make objectively true or false assertions. Finally, in his late article "Thoughts," Frege uses explicitly the term "third realm" for a realm or world of objects that are neither physical nor mental.

Let me quote a few of the most direct statements Frege makes on this topic. On numbers as logical objects, he writes in *Foundations*:

Yet surely the number one looks like a definite particular object, with properties that can be specified, for example that of remaining unchanged when multiplied by itself? (Introduction, p. ii)[4]

But, it will perhaps be objected, even if the Earth is really not imaginable, it is at any rate an external thing, occupying a definite place; but where is the number 4? It is neither outside us nor within us. And, taking those words in their spatial sense, that is quite correct. To give spatial co-ordinates for the number 4 makes

no sense; but the only conclusion to be drawn from that is, that 4 is not a spatial object, not that it is not an object at all. Not every object has a place. (§61, p. 72)

On the existence and objectivity of such objects:

For number is no whit more an object of psychology or a product of mental processes than, let us say, the North Sea is. . . . [It] is something objective. (§26, p. 34)

[E]ven the mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there and give it a name. (§96, pp. 107–8)

And finally in "Thoughts":

A third realm must be recognized. Anything belonging to this realm has it in common with ideas that it cannot be perceived by the senses, but has it in common with things that it does not need an owner so as to belong to the contents of his consciousness. (p. 363)[5]

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What is my goal now? Is it not clear, on the basis of these quotations, that Frege is a Platonist? Do I want to deny that and to reinterpret him as a non-Platonist? No, my goal is rather the following: I want to show that quotations such as those above, as well as the metaphysical picture of Platonism they conjure up, have to be treated with care. That is to say, they require a careful, critical interpretation. Without such an interpretation—merely relying on the vague picture of a "Platonic heaven"—we are dealing with "naïve Platonism" (or, respectively, with "naïve anti-Platonism"). My main goal is to go beyond the corresponding naïveté.

I am not the first person to urge for care in this connection. Christian Thiel and, to some degree, Michael Dummett urged us already in the nineteen-sixties and early seventies not to rely too much on the picture of a "Platonic heaven."[6] Since then Gottfried Gabriel has talked repeatedly about a "Platonic hypostatization" of abstract objects and has objected to a corresponding interpretation of Frege.[7] And recently a variety of interpreters, especially in the English-speaking world, have begun to reinterpret Frege's philosophical approach in such a way that a refined, non-naïve understanding of Frege's Platonism is slowly beginning to emerge.[8] But what exactly does this new understanding consist in; or, more generally, what alternatives are there in this connection? That is exactly what I want to make explicit and to clarify further in what follows.

The clarification I am interested in has to do with questions about how to understand notions such as "object," "reference," "truth," "objectivity," and so forth or, better, how to explain and relate them to each other. In other words, it has to do with the question of what philosophical role is played by claims such as these: that numbers are "logical objects," that we "refer" to them with our number words, and that we make "true and false assertions" about them.

My main thesis in this connection is that the Platonic picture we encountered above does not determine this role uniquely. Rather, it leaves room for two very different interpretations and thus for two different kinds of Platonism. For ease of reference, let me introduce names: I will call them "Platonism A" and "Platonism B," respectively.[9]

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First to Platonism A: The starting point, or at least the background, for this kind of Platonism is a realistic understanding of the physical world, the world of objects such as tables, chairs, the Eiffel Tower, and the Moon. The Platonist picture above is now interpreted as asserting that numbers are objects "in the same sense"—that they exist, are independent, have determinate properties, and so forth "in the same sense"—except that they do not inhabit the physical world, nor anyone's mental world, but a separate world of "abstract objects." Such an understanding of the notions of "object," "existence," and so on is the basic ontological aspect of Platonism A.

Understood as such, individual numbers can now directly be given names: "the number one," "the number two," and so forth. Similarly we can talk directly about various properties of numbers: about the property "to be even," "to be a prime number," and so on. This is the basic semantic aspect of Platonism A. Furthermore, on that basis we can then explain the truth (or falsity) and the objectivity of arithmetic statements. For example, the statement "The number four is even" is now objectively true in the sense that the object to which we refer with "the number four" really does have the property meant by "is even." This explanation of truth and objectivity is the basic metaphysical aspect of Platonism A.

Let me make even clearer and more explicit what, for my purposes, is crucial here. Let us start again with the basic ontological aspect. In Platonism A, as I have just described it, the notions of "object," "objective existence," and "determinate property" are presupposed as primitive; they are fundamental or primary notions—notions that are not really explained themselves, but presupposed and used in other explanations. They are introduced, or at least illustrated and motivated, by appealing to physical examples, such as the object called "Eiffel Tower," its existence in Paris, its height, weight, and so forth. Implicitly it is presupposed here that such objects are "independent" and "determinate in themselves" in the sense that their existence and their properties (and thus the causal roles they play) do not depend on our existence as observers, our thinking, knowledge, and so on. Put briefly, we start from a notion of "objecthood" that is inspired by the paradigmatic example of physical objects (understood in a realist sense).

As far as the basic semantic aspect of Platonism A is concerned, it should furthermore be noted that the naming relation, or the notion of "reference," is considered to be primitive as well, or at least to be fundamental for the further explanations given.[10] The notion of truth can then be explained in a substantive way; in particular, truth is explained as correspondence. And this also leads to a corresponding notion of objectivity. Altogether an arithmetic statement is now

objectively true or false in the sense that we can "measure" it against an independent, abstract world of numbers—analogously to how we "measure" a physical statement against an independent, realistically understood physical world.[11] Here it is important, to emphasize again, that this is meant to be a substantive explanation of truth and objectivity. The role of such a substantive explanation is to give Platonism A philosophical "weight" or "bite."[12]

The question Platonism A provokes immediately is how we can ever have access to such a world of abstract objects. In the physical case this seems to be relatively unproblematic since we are in causal contact with the corresponding objects. But the "abstractness" of an object such as the number two implies that we are precisely not in causal contact with it. For the same reason the postulation of a corresponding kind of quasi-visual "perception," as it is sometimes attributed to Platonism, is immediately problematic as well. For how is such a perception of abstract objects—a "sixth sense," so to speak—supposed to work if not as a causal contact in the end? That seems completely unclear. The epistemological side of Platonism A is, then, really a problem.[13]

This problem can perhaps best be dramatized by considering the possibility of a "fundamental mistake." Let us start again with the parallel to the physical world. It is clear that we can be fundamentally mistaken about physical objects. For example, it is possible that much, perhaps everything, we think we know currently about a distant planet can turn out to be false. It is even possible that the postulated planet does not exist at all. Similarly a whole physical theory can turn out to be false. The radical possibility opened up by Platonism A is now this: The same could be the case for our ordinary arithmetic! That is to say, perhaps there are no natural numbers after all; or perhaps their properties are fundamentally different from what we have assumed so far? Note that this is possible even if our ordinary arithmetic is consistent.

Hardly anyone assumes, of course, that this is in fact the case. However, the possibility opens up at least in principle because Platonism A explains the objective truth of arithmetic statements as correspondence to a completely separate world of abstract objects. Whether such a correspondence holds or not is, thus, conceptually independent from our usual practice of judging and inferring. That is exactly what opens up the possibility of a fundamental mistake: our arithmetic judgments and inferences can be wholly in order internally—consistent, systematic, and so forth—but still wrong in the sense of a missing or at least incomplete correspondence to the postulated world of abstract objects.

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Is Frege a representative of Platonism A? It may initially appear so, especially if the remarks quoted above are considered in and of themselves. On the other hand, we can quickly make a few observations that call such an interpretation into question. Recall once more the three basic aspects of Platonism A as just discussed: (i) a certain way of understanding "objecthood," guided by the example of physical objects; (ii) the explanation of truth and objectivity as correspondence, based on such "objecthood" and on a related notion of "reference"; and (iii) the resulting problem of access, or the possibility of a "fundamental mistake," even if our normal arithmetic is consistent. With respect to all three aspects Frege's position differs from Platonism A, as I will now show.

Let us start with the problem of access, understood in the sense above. A first, somewhat indirect observation here is that Frege seems to have no sense for this problem—no corresponding doubt plays any role in his writings. Moreover, he does not postulate any kind of quasi-visual perception of logical objects, constituting a "sixth sense." He is even directly opposed to a corresponding point of view, as the following passage from *Foundations* shows:

> In arithmetic we are not concerned with objects which we come to know as something alien from without through the medium of the senses, but with objects given directly to our reason and, as its nearest kin, utterly transparent to it. (§105, p. 115)

The reference to "reason" in this passage is, no doubt, in need of clarification. The passage makes clear, however, that for Frege the access to logical objects, in particular to numbers, does not consist of a perception analogous to using our normal five senses. In the last respect it consists, instead, of the thinking of corresponding thoughts—or, better, of the corresponding judgments and inferences, as well as of our logical reflection on these judgments and inferences.[14] That is, in any case, exactly what Frege does: reflect on our normal arithmetic judging and inferring, and reconstruct it logically (more on that later).

As far as the possibility of a fundamental mistake is concerned, there is, indeed, one danger explicitly recognized by Frege for his logical reconstruction of arithmetic (a danger that will eventually make him give up this reconstruction): its inconsistency. Note, however, that this has to do with an "internal" problem for this reconstruction, not with a missing "external" correspondence to a Platonist world, in the sense of Platonism A. On the other hand, Frege is not only concerned about the inner consistency of his new system. He also wants to provide our ordinary arithmetic judgments and inferences with a new foundation. In other words, Frege's approach is meant to lead to a recovery of normal arithmetic judgments such as "2 + 2 = 4." Then again, that involves, once more, only "inner"—systematic, logical—issues.

But does Frege not explicitly recognize the objective truth or falsity of arithmetic judgments? And does that not inevitably lead, explicitly or implicitly, to an explanation of such objective truth in the sense of Platonism A? In response to that question two points can be made: First, Frege rejects repeatedly—in its most extended form in the late article "Thoughts," but more briefly already earlier—not only a correspondence explanation of truth, but any explanation of truth at all.[15] Put briefly, for him the truth (or falsity) of our judgments, especially our logical judgments, is fundamental and not further reducible. But that contradicts, then, a main aspect of Platonism A.

Second, Frege also does not explain the objectivity of arithmetic judgments as discussed so far. That is not to say that he denies their objectivity. Rather, he writes:

> It is in this way that I understand objective to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories of earlier sensations, but not what is independent of the reason,—for what are things independent of the reason? To answer that would be as much as to judge without judging, or to wash the fur without wetting it. (*Foundations*, §26, p. 36)

A bit less metaphorically:

> What is objective . . . is what is subject to laws, what can be conceived and judged, what is expressible in words. (§26, p. 35)

And especially:

> My definition [of number] lifts the matter onto a new plane; it is no longer a question of what is subjectively possible but of what is objectively definite. For in literal fact, that one proposition follows from certain others is something objective, something independent of the laws that govern the movements of our attention, and something to which it is immaterial whether we actually draw the conclusion or not. (§80, p. 93)

Crucial for my purposes is that Frege, in passages such as these, connects the notion of objectivity with the possibility of judging, inferring, and so forth, and not with the external correspondence to an independent Platonist world.[16]

Finally, what about the "objecthood" of numbers? Are Frege's views in that connection not a clear reason to regard him as a representative of Platonism A? Here, too, there are some passages that should give us pause immediately. As a prime example, consider the following remark by Frege on the "self-subsistence" of numbers:

> The self-subsistence which I am claiming for number is not to be taken to mean that a number word signifies something when removed from the context of a proposition, but only to preclude the use of such words as predicates or attributes, which appreciably alters their meaning. (§60, p. 72)

Thus for Frege it is important how number words are used "in the context of a proposition," namely as object words, not as "predicates" or "attributes." That is what the "self-subsistence," and more generally the "objecthood," of numbers involves, not a direct comparison to the way in which physical objects exist.

The observations I have made so far could be extended further. In particular, one could say more about the way in which Frege connects the notions of "reference," "truth," and our usual practice of judging and inferring conceptually. His corresponding approach is in general very different from Platonism A, in many respects even contrary to it.[17] Also, his notion of "object," as well as the correlative notion of "concept," is more general than, and understood differently from, the corresponding notions in Platonism A. For Frege these are primarily logical, not naïvely ontological notions.[18] But considerations such as these will only be convincing in the end if we reinterpret Frege's position as a whole. What we need, in other words, is an alternative to the interpretation so far. That is what I want to turn to now.

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It is at this point that it becomes crucial to look at Frege's Platonist remarks "in context," especially in the context of his general approach in *Foundations* and *Basic Laws*. How does Frege approach, methodologically, the question of what numbers are—in particular whether they are logical objects or not? His method has two sides that complement each other: On the one hand, he examines our normal arithmetic judgments and inferences informally concerning the issue of which content and which inferential structure they have. This provides him with the "raw material" for his further steps. On the other hand, he develops a systematic reconstruction of this raw material within the framework of his new logical system.

As far as Frege's informal analysis of our normal arithmetic judgments and inferences is concerned, the results most important for present purposes can be summarized briefly as follows: First, we misunderstand these judgments and inferences if we interpret them in a formalist, empiricist, or psychologistic manner, as some of Frege's contemporaries do. In other words, a statement like "2 + 2 = 4"—understood as part of our system of arithmetic judgments and inferences—does not have to do, either with respect to its content or with respect to its justification, with the properties of numerals, with empirical observations and experiments, or with mental objects and psychological processes. Instead all such numerical statements have to do with concepts, as well as with the logical consequences resulting from that connection to concepts. This is the core of Frege's logicism.

The objectivity of arithmetic statements is, thus, intimately connected with the objectivity of judgments and inferences concerning concepts, in particular logical concepts. But that means that it is based on the objectivity of logical judgments and inferences. It is in this sense that Frege writes (as already quoted above):

> My definition [of number] lifts the matter onto a new plane; it is no longer a question of what is subjectively possible but of what is objectively definite. For in literal fact, that one proposition follows from certain others is something objective, something independent of the laws that govern the movements of our attention, and something to which it is immaterial whether we actually draw the conclusion or not. (*Foundations*, §80, p. 93)

Crucial for my alternative interpretation is now the following thesis: In the end, Frege starts directly from the objectivity of our logical judgments and inferences. That is to say, their objectivity is a basic assumption; it is fixed and not further reducible. All other explanations already presuppose and build on it, not the other way around (as in Platonism A). This is the basis for Frege's Platonism.

A more fine-grained analysis of our arithmetic judgments and inferences leads Frege to two additional results: First, all numerical statements do not just have to do with concepts in general; rather, the relation of one-to-one mappability of two concepts, or, better, of their corresponding extensions, onto each other is crucial. Second, numerical statements can always be analyzed in such a way

that they contain number words or numerals as playing the role of objects names; for example, the statement "Jupiter has four moons" can be analyzed as "The number of moons of Jupiter = 4." What that means is that number words and numerals play a certain logical role.

This brings us to Frege's logic, the new system he presents for the first time in *Begriffsschrift*, presupposes implicitly in *Foundations*, and develops further, as well as applies explicitly to arithmetic, in *Basic Laws*. Fundamental for this logic is the distinction between function and argument and, building on it, the distinction between function and concept names, on the one hand, and object names, on the other. This distinction replaces the earlier Aristotelian distinction between subject and predicate. In the end Frege's new distinction has to do with insights into how our entire informal system of judgments and inferences in all the sciences together can be analyzed. Frege assumes that he has found the best—or even the only correct—analytic approach to it.

If we apply Frege's logic to judgments and inferences concerning physical objects, we can see that expressions such as "the Eiffel Tower" or "the Moon" function in them as object names—in Frege's logical sense. Insofar as this is the case, the naïve ontological notion of object from above matches Frege's logical notion of object. But Frege's notion is broader, and it has a different basis. When applied to arithmetic judgments and inferences, in particular, we can see that expressions such as "the number four" also play, logically speaking, the role of object names, not that of function or concept names. That is the background not only for Frege's remarks about the self-subsistence of numbers quoted above, but also for his more general thesis that numbers are objects.

In itself this is not sufficient, however, to be able to speak of numbers as objectively existing logical objects. Two things have to be added: first, it has to be determined that all statements in which the corresponding expressions occur are objectively true or false; second, it has to be clarified what this truth or falsity is based on, namely purely logical foundations. Now, both of these things together are exactly what Frege's logical reconstruction of arithmetic is supposed to accomplish. Its explicit goal is to reduce all arithmetic judgments to basic logical laws and definitions, thus providing a foundation for their objective truth or falsity, among others. If this goal can be achieved, then Frege has, in the terminology of *Foundations*, given all arithmetic statements "sense" (and thus, in the more refined terminology of *Basic Laws*, "sense" and "reference").[19]

At the same time, for Frege this is also exactly what gives us "access" to the natural numbers and their properties. It is in this sense that he writes in *Foundations*:

> How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs. (§62, p. 73)

Similarly a little later:

> Now for every object there is one type of proposition which must have a sense, namely the recognition-statements, which in the case of numbers is called an identity. (§106, p. 116)

In other words, Frege's explanation of what numbers are, which properties they have, how we have access to them, and so forth, consists exactly in reducing arithmetic statements, especially identities, to the basic laws and definitions of his logical system.[20]

As sketched in *Foundations* and worked out in detail in *Basic Laws*, Frege's logical reconstruction of arithmetic takes on a specific (and problematic since inconsistent) form in the end, based on his definition of the natural numbers as extensions of concepts. This definition is motivated by two observations already mentioned above: first, that numerical statements have to do with concepts, extensions of concepts, and the relation of one-to-one mappability; second, that from a logical point of view numerical expressions, just like names for extensions of concepts, have to be regarded as object names. For my purposes only the following is important here: the answer to the question of what numbers are is thus reduced for Frege to the answer to the question of what extensions of concepts are.

The answer to the latter question is, in *Foundations* (somewhat vaguely and imprecisely) as well as in *Basic Laws* (more definitely and explicitly), this: extensions are logical objects. And what does that mean? According to my interpretation, now applied to extensions of concept, the following: The names with which we refer to extensions play the logical role of object names, not the role of function or concept names; as such they are used in judgments and inferences that are objectively true or false, valid or invalid; and all these judgments and inferences are based, fundamentally, on assumptions that have a purely logical nature. (The first two points make extensions into self-subsisting, objectively existing objects; the third point makes them into logical objects.) Finally, in the background we have again this assumption: the objectivity of the basic logical laws on which the whole system is founded is a fundamental given. This is what Frege's Platonism concerning extensions, thus also concerning numbers, amounts to in the end.

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So far I have taken a new look at Frege's Platonism within the context of his general approach and starting from a few important quotations. The kind of Platonism we arrived at this way—Platonism B, in contrast to Platonism A—can also be described more directly. I want to add such a description now.

The main difference between Platonism B and Platonism A consists in the fact that in Platonism B we do not simply start with the postulation of a world of abstract objects in analogy to the world of physical objects. Rather, we start from a consideration of the corresponding judgments and inferences, here arithmetic judgments and inferences. We then analyze, or reconstruct, the whole system of these judgments and inferences in such a way that the following becomes clear: the judgments are determined, in an objective way, with respect to their truth values, the inferences with respect to their validity; this is done in such a way that the whole system is based, in the end, only on fundamental logical assumptions, not on empirical or other assumptions; and it is done in

such a way that the expressions used to refer to numbers systematically play the role of object names, not that of function or concept names. If all of this is the case, then what is determined is not only the logical category of numbers as objects, but also everything that holds with respect to them; and it is determined objectively, in the sense that it is based completely on basic logical laws.

The fundamental idea behind such a view is the following: What counts with respect to "abstract" objects such as the natural numbers is, at bottom, which corresponding judgments and inferences are true or valid, respectively. If that is fixed in an objective way—and so that all the usual arithmetic truths such as "$2+2=4$" follow—then numbers are determined in themselves. If it is fixed so that number words play the logical role of object names, then numbers are self-subsistent objects. And if it is fixed purely logically, then they are logical objects. What is crucial for the "objecthood" of numbers is, thus, their objective determinateness as objects via basic logical laws. One can even say that that is all their "objecthood" amounts to.

At this point the following objection may suggest itself: Is this really a kind of Platonism? Is it not rather a refined kind of formalism, fictionalism, or linguistic idealism? Two things can be said in response: First, for a representative of Platonism B—Frege as I interpret him—arithmetic judgments and inferences are still objectively true or false, valid or invalid. This is based on the fact that the basic logical laws from which we start are considered to be objectively true or valid, respectively. These logical laws are thus not just empty formulas or merely fictional, linguistic conventions.[21] Second, if the existence of logical objects, in particular numbers, can be derived from these basic logical laws—that is, if the corresponding existentially quantified propositions turn out to be objectively true—that means that numbers exist (also that we can refer to them, can make true or false statements about them, and so forth). Crucial here is that "existence"—similarly to "objecthood" above—is now understood in terms of the objective determinateness of the corresponding judgments, especially existentially quantified judgments.

Does that mean that, along these lines, numbers are objects "in the same sense" as physical objects? Yes and no. Yes insofar as the "objecthood" of the Eiffel Tower or of the Earth can also be understood in terms of the determinateness of the corresponding judgments and inferences. No insofar as the Eiffel Tower and the Earth are in addition, to use Frege's terminology, "actual" objects. That is to say, they can be located in space and time, they stand in causal relation to other physical objects and corresponding processes, and so forth. The latter does not hold for logical objects, as Frege emphasizes. Once more, in Platonism B we work with a general logical (not a naïve ontological) notion of object, one under which logical as well as physical objects fall, as two special cases. From this point of view the essential difference between the two cases consists of the difference in what the determinateness of the corresponding judgments and inferences is based on.[22]

To repeat, the basic idea in Platonism B is the objective determinateness of numbers as logical objects (in Frege's sense). Let me add two more remarks to round off my discussion of such an approach. The first concerns the question of consistency. It should be pretty clear that "objective determinateness" includes the consistency of the system of corresponding judgments and inferences—since their inconsistency would imply that all judgments are both true and false, so exactly not determinate as the one or the other. This explains, among others, why consistency is so important for Frege. Second, the way in which numbers are understood to be logical objects in Platonism B is not necessarily connected with Frege's specific reduction of numbers to extensions of concepts. His reduction is, rather, just one of potentially many ways in which arithmetic judgments and inferences can be determinate with respect to their truth value and validity. At the same time, Frege himself considered his reduction apparently to be the best or "right" one—at least until Russell informed him of the inconsistency of his system.

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Let me come to a close. My goal in this paper was to argue for two main theses: First, there is not just one way to understand "Platonist" statements of the kind mentioned initially, including the statement that numbers are abstract or, better, logical objects. Insofar as this is the case, the naïve metaphysical picture of a separate "Platonist heaven" in which such objects exist is vague and ambiguous. One can interpret it in the sense of Platonism A, starting with the analogy to the physical world, a corresponding notion of "objecthood," and the explanation of truth and objectivity as correspondence. Or one can understand it in the sense of Platonism B, in terms of a logical notion of "objecthood," a corresponding analysis of the relevant judgments and inferences with respect to their determinateness, and starting with the objectivity of the corresponding logical laws. These amount to two very different positions, it seems to me.[23]

Second, Frege's position is a version of Platonism B, not of Platonism A. My reasons for such an interpretation arose out of considering Frege's corresponding views in the context of his general approach, both in *Foundations* and in *Basic Laws.* Looking back now, my interpretation allows us to make sense of several Fregean remarks that are hard to understand otherwise—especially the remarks in which he talks about the "self-subsistence" of numbers, those in which he describes their "objectivity," and those in which he explicitly denies an explanation of truth in terms of correspondence. Finally, such an interpretation makes intelligible why Frege himself did not see a problem with our access to the "realm" of logical objects, while the discovery of the inconsistency of his system hit him hard.

A final clarification: As has probably become clear, for me Platonism B is more attractive than Platonism A. At the same time, I do not think that Platonism B is entirely unproblematic—either as a position in itself or as an interpretation of Frege. In particular, the following three main problems or questions arise directly out of my discussion: (1) How exactly are we supposed to understand the objectivity of the logical laws on which everything depends for Frege? And is his assumption of their objectivity defensible in the end (perhaps in a weakened form)? Related to that, in which sense is Frege's logical system for him "the right one"?[24] (2) Why, or in which sense, does Frege consider his specific reduction of the natural numbers to be "the right one"? This is

especially interesting if we compare it to other such reductions (Russell's and Zermelo's, say). More generally, in which sense, or to what degree, is a reduction of arithmetic to logic (or type theory, set theory, and so forth) defensible at all?[25] Finally, of course: (3) Is it possible to develop a position that upholds most of Frege's basic Platonist theses, but does not fall pray to Russell's antinomy (and similar problems)? In other words, is there a consistent variant of Platonism B?[26]

It is certainly not easy, in some respects perhaps impossible, to find completely satisfying answers to these questions. Nevertheless, they seem to me to be worth investigating further—both as far as a better understanding of Frege is concerned and as far as the philosophy of mathematics more generally is concerned.[27] φ

Notes

[1] In this paper I develop further ideas already presented in *Frege, Wittgenstein, and Platonism in Mathematics* and in "Frege's Influence on Wittgenstein: Reversing Metaphysics via the Context Principle." Unlike earlier, however, here I concentrate exclusively on Frege's Platonism. My approach is strongly influenced by Thomas Ricketts, "Objectivity and Objecthood: Frege's Metaphysics of Judgment" (as far as Frege is concerned), and by W. W. Tait, "Truth and Proof: The Platonism of Mathematics" (as far as Platonism is concerned). In W. P. Mendonça & P. Stekeler-Weithofer, "Was Frege a Platonist?" one can find a comparable interpretation of Frege, but with the focus on Fregean "thoughts."

[2] Stephen Barker, "Number," in *Encyclopedia of Philosophy*, edited by P. Edwards, pp. 526–30. Many similar characterizations of "Platonism" can be found elsewhere. For a subtler, more differentiated characterization, see the corresponding entry, written by G. Gabriel & T. Rentsch, in *Historisches Wörterbuch der Philosophie*, edited by J. Ritter & K. Gründer, vol. 7, P–Q, pp. 985–88.

[3] See P. Benacerraf, "Mathematical Truth," and P. Kitcher, "The Plight of the Platonist."

[4] My page numbers are based on the English editions of these works listed in the bibliography. In general I will use the standard English translations, although occasionally modified in minor ways by me.

[5] This last quotation concerns directly only "thoughts." Nevertheless it seemed appropriate here since it illustrates how Frege is willing to talk about non-physical, non-mental entities.

[6] See C. Thiel, *Sinn und Bedeutung in der Logik Gottlob Freges*, and M. Dummett, *Frege: Philosophy of Language*; compare also Dummett, *Frege: Philosophy of Mathematics*.

[7] See G. Gabriel, "Leo Sachse, Herbart, Frege und die Grundlagen der Arithmetik," as well as Gabriel's earlier articles on Frege mentioned in it.

[8] I have in mind, in particular, recent work by Cora Diamond, Thomas Ricketts, and Joan Weiner; see especially the chapters on Frege in Diamond's *The Realistic Spirit*, Ricketts's "Objectivity and Objecthood" and "Logic and Truth in Frege," and Weiner's *Frege in Perspective*. Compare also Crispin Wright's neo-Fregean investigations, starting with *Frege's Conception of Numbers as Objects*.

[9] This corresponds to my earlier distinction between "metaphysical" and "contextual Platonism"; see Reck, "Frege's Influence on Wittgenstein." One could also talk about "extreme" versus "moderate," "ontological" versus "logical," or "object" versus "law Platonism" (see below for motivation for these labels). But I want to use more neutral terms here, at least initially.

[10] In her discussion of an "un-Fregean" kind of Platonism in *Frege in Perspective*, Joan Weiner emphasizes this aspect as well; see especially chapter 5, "Platonism: Fregean and UnFregean."

[11] Steve Gerrard emphasizes and discusses this aspect of Platonism (and Wittgenstein's corresponding reaction) in "Wittgenstein's Philosophies of Mathematics."

[12] Such a Platonist explanation in terms of correspondence is also often taken to be an explanation of the truth of logical laws, especially of the law of the excluded middle. Compare here several of the papers in Michael Dummett's *Truth and Other Enigmas*.

[13] I think that the notions of "object" and "reference" as used in Platonism A are also problematic in the end. However, I want to leave the corresponding problems aside here; see Tait, "Truth and Proof," and Reck, "Frege's Influence on Wittgenstein," in this connection.

[14] Thus for Frege the access to logical objects does not consist of a kind of "knowledge of" ("knowledge by acquaintance"), as the perception of physical objects by means of our usual five senses is often understood, but of a kind of "knowledge that." Frege's Platonism differs in this respect strongly from the Platonism of the early Russell.

[15] As Frege argues, any such explanation of the notion of truth already presupposes that notion and is circular. See here especially "Thoughts," pp. 352–53; compare also Thomas Ricketts's interpretation of the corresponding "regress argument" in "Objectivity and Objecthood" and "Logic and Truth in Frege."

[16] Compare, however (as Marco Ruffino has reminded me), the conflicting interpretation of Frege's remarks on "reason" in Tyler Burge, "Frege on Knowing the Third Realm." My goal in this part of the article is not to refute interpretations such as Burge's conclusively, but only to motivate the search for an alternative, one I will sketch further below.

[17] This has to do with Frege's "context principle" ("never ask for the meaning of a word in isolation, but only in the context of a sentence" [*Foundations*, introduction, p. x]); compare here Reck, "Frege's Influence on Wittgenstein," in particular what I call "a reversed order of explanation" in it.

[18] As Thomas Ricketts puts it: "[For Frege] ontological categories are wholly supervenient on logical ones"; see "Objectivity and Objecthood," p. 66ff.

[19] In *Foundations* Frege has not yet distinguished clearly between "sense" and "reference." But the interpretation just given can, as indicated in the text, easily be translated into the later terminology. More generally, I do not think that with respect to the issues crucial for me—especially with respect to Frege's general approach—there is an important difference between *Foundations* and *Basic Laws*.

[20] It is important here to conceive of Frege's system in such a way that contents are expressed in it. In particular, Frege assumes that the logical laws with which he starts do have a definite content (in the end, "sense" and "reference"). Thus they are not part of an object language to be understood in a purely formalist way, that is, as being provided with "content" after the fact via various interpretations. Compare here Ricketts's "Objectivity and Objecthood: Frege's Metaphysics of Judgment" as well as Warren Goldfarb's more recent "Frege's Conception of Logic."

[21] Compare here endnote 20 and the references in it.

[22] Frege's third case (in his way of counting, the second) of psychological or mental "objects" is different insofar as in this case we are only dealing with subjective determinateness. More generally, the corresponding three ways of being determinate—objective-empirical (especially spatiotemporal and causal), objective-logical, and subjective—are, in the end, what underlies Frege's talk of "three realms."

[23] It may be tempting, again, to think that only Platonism A deserves the name "Platonism" in the end, especially if one assumes that explaining the objectivity of logic and arithmetic in terms of their correspondence to a "world behind" necessarily belongs to such a position. However, this seems to me to be a tendentious and unnecessary restriction of the use of "Platonism." In any case, I hope that, independently from questions about an adequate terminology (see here endnote 9), the difference between the two positions discussed by me has become clear.

[24] It is at this point that some interpreters suspect a version of "strong" Platonism in Frege; for example, Tyler Burge writes in "Frege on Knowing the Third Realm," p. 645, footnote 16, "Frege sees the whole logical structure, not just objects, in a Platonic fashion." However, what exactly is meant by "in a Platonic fashion" here is not clear immediately (especially if one rejects Platonism A as an interpretation). A very different alternative consists in interpreting Frege's logic as a "transcendental condition" for all our judging and inferring, in a Kantian or neo-Kantian sense. But what exactly that means, as well as whether it really corresponds to

Frege's views, is not immediately clear either. As a weakened variant of the latter, one can also appeal to the "indispensability" of Frege's logic (or some equivalent theory) for the natural sciences (see Quine's "On What There Is"). However, Frege's strong anti-empiricism makes such a variant not plausible as an interpretation of his position.

[25] Compare here Richard Dedekind's non-reductive approach in "The Meaning and Nature of Numbers," as discussed in my article "Dedekind's Structuralism: An Interpretation and Partial Defense."

[26] Compare here the "neo-Fregean" approach in Crispin Wright's *Frege's Conception of Numbers as Objects*, developed further in Hale & Wright, *The Reason's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*. See also George Boolos's contributions to the discussion "how to save Frege from contradiction" in his *Logic, Logic, and Logic* as well as the corresponding articles in William Demopoulos's *Frege's Philosophy of Mathematics*.

[27] This paper was originally written for the conference *Gottlob Frege: Werk und Wirkung*, University of Jena, Germany, November 4–7, 1998. A German version of it, entitled "Freges Platonismus im Kontext," was published in *Gottlob Frege: Werk und Wirkung*, ed. G. Gabriel and U. Dathe (Paderborn: Mentis, 2000), pp. 71–89. I would again like to thank Gottfried Gabriel and the other organizers for the invitation to the Jena conference, and I am grateful to the participants for helpful and encouraging comments, especially Peter Simons, Marco Ruffino, Wolfgang Kienzler, Uwe Dathe, and Bernd Buldt.

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