From: THE CAMBRIDGE COMPANION TO CARNAP, M. Friedman & R. Creath, eds., 2007

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8 Carnap and modern logic

A distinguishing feature of analytic philosophy, or at least of one central strand in it, is the use of modern logic for the purpose of clarifying and solving philosophical problems. The most prominent figure in this tradition was Bertrand Russell; and second only to Russell was Rudolf Carnap. Directly and strongly influenced by Russell, Carnap passed on this influence to legions of later philosophers, including such widely influential figures as W. V. O. Quine. It is well known that Carnap was a main expositor and promoter of modern logic, as illustrated by his textbooks on the subject, from *Abriss der Logistik* (1929) to *Einführung in die symbolische Logik* (1954). It is also well known that Carnap applied logic substantively, both in his own constructive endeavors in philosophy and in his criticism of metaphysics, as in *Der logische Aufbau der Welt* (1928a), "Überwindung der Metaphysik durch logische Analyse der Sprache" (1932d), and *Logische Syntax der Sprache* (1934c).

Less well known is the fact that, in addition, Carnap was actively engaged in research on pure logic and related questions in early metamathematics. In particular, during large parts of the 1920s – parallel and subsequent to his work on the *Aufbau* – Carnap was pursuing a major research project in this area. A main goal of this project was to

[1] In the present chapter I draw on a series of recent writings on Carnap (Bonk and Mosterin, 2000; Awodey and Carus, 2001; Awodey and Reck, 2002a; Awodey and Reck, 2002b; Goldfarb, 2003; Reck, 2004; Reck and Awodey, 2004; and Goldfarb, 2005). There are three ways in which I will try to advance the discussion: first, by emphasizing the uniqueness of the position Carnap occupied in the history of logic (sections I and II); second, by sharpening the focus on the notion of logical consequence (sections II and III); and third, by making explicit connections to Carnap's interests in the notion of mathematical truth (sections III and IV).

combine, and to reconcile, the approaches to logic and the foundations of mathematics he had encountered in interactions with Gottlob Frege and Bertrand Russell, on the one hand, and in the works of David Hilbert and his followers, on the other. Carnap's project also had direct connections to contemporary work by Abraham Fraenkel on axiomatics, Kurt Gödel on incompleteness, and Alfred Tarski on the foundations of meta-logical notions. Carnap was at the cutting edge of research in modern logic during this period, both in terms of his personal contacts and his own endeavors. While these endeavors did not bear the systematic fruits he initially envisioned, they did lead to some partial results; they also had a significant influence at the time. It is this contribution by Carnap – a long-neglected side of his career – to which I want to introduce the reader in the present chapter.

In the first section of the chapter, Carnap will be introduced as a student of modern logic. This will include a brief account of the influence Frege and Russell had on him; but I will also describe his early interest in the axiomatic method, especially in Hilbert's work. In the second section, we will see how Carnap, attempting to synthesize these two major influences, was led to a project in "general axiomatics." He was not the only person to be led in that direction, as a look at related work by Fraenkel will illustrate. In the third section, Carnap's project will be discussed in more detail, focusing on a book manuscript, *Untersuchungen zur Allgemeinen Axiomatik* (ASP RC 080-various), left unpublished by him, but recently edited and made available in print (2000). This discussion will make explicit some inherent limitations of, or problems with, Carnap's approach. Recognition of these problems caused him to abandon the project around 1930 – but not without first having influenced Gödel and formed the basis for some interactions with Tarski. While it may appear, at that point, that Carnap's 1920s project was mostly a failure, in the final section I will point out its interesting aftermath and continuing significance.

I. CARNAP AS A STUDENT OF MODERN LOGIC

In his "Intellectual Autobiography" (1963a, 3ff.) Carnap tells us that upon entering the University of Jena in 1910 his main interests were first in philosophy and mathematics, then in philosophy and physics.

For a brief period he tried his hand at experimental work in physics, but he quickly turned towards more theoretical issues, including Kant's views about space and time and their relation to recent developments in physics. Early on, Carnap also attended three classes by Frege on logic and the foundations of mathematics: "Begriffsschrift I" (1910–11), "Begriffsschrift II" (1913), and "Logik in der Mathematik" (1913–14).[2] In these classes Carnap was introduced to modern logic, as originating in Frege's and Russell's works. This was quite unusual – not only were very few classes on modern logic taught anywhere at the time, Frege's particular classes, while offered regularly, were also attended by very few students.

While Carnap found Frege's classes fascinating, he didn't recognize the full significance of the logic he encountered in them right away, especially not its potential fruitfulness in addressing philosophical problems. As he explained later:

> Although Frege gave quite a number of examples of interesting applications of his symbolism in mathematics, he usually did not discuss general philosophical problems. It is evident from his works that he saw the great philosophical importance of the new instrument which he had created, but he did not convey a clear impression of this to his students. Thus, although I was intensely interested in his system of logic, I was not aware at that time of its great philosophical significance. Only much later, after the first world war, when I read Frege's and Russell's books with greater attention, did I recognize the value of Frege's work not only for the foundations of mathematics, but for philosophy in general. (1963a, 6)

In addition, Carnap's attention was soon diverted by the outbreak of the First World War, which he experienced as an "incomprehensible catastrophe" (1963a, 9) and which took him away from the University of Jena as a soldier in the German army.

It was only after coming back from the war that Carnap could take up his academic interests again. In 1919, he started to study Whitehead and Russell's *Principia Mathematica* (1910–13). Frege had mentioned this work in his classes and, on the basis of what he had already learned, Carnap was able to assimilate its content by himself. Through Frege's influence, he was thus part of the first generation of thinkers on which *Principia* had an impact. From 1920 on, he also

[2] For Carnap's own notes from these classes, see Reck and Awodey (2004). For further discussion, see Reck (2004); but note that there are some inaccuracies in the corresponding dates given in that article.

returned to Frege's own writings and studied them carefully, especially *Die Grundgesetze der Arithmetik* (1893/1903), Frege's *magnum opus*. Later Carnap described the effect of these studies as follows:

> I began to apply symbolic notation, now more frequently in the *Principia* form than in Frege's, in my own thinking about philosophical problems or in the formulation of axiom systems. When I considered a concept or a proposition occurring in a scientific or philosophical discussion, I thought that I understood it clearly only if I felt that I could express it, if I wanted to, in symbolic language. I performed the actual symbolization, of course, only in special cases where it seemed necessary or useful. (1963a, 11)

Notice that, in direct connection with assimilating Frege's and Russell's works, Carnap mentions the goal of applying their logic "in the formulation of axiom systems." Carnap's first project for a dissertation, entitled "Axiomatic Foundations of Kinematics," stems from the same period. However, the two people at the University of Jena to whom he showed his proposal – the physicist Max Wien and the philosopher Bruno Bauch – both rejected it. Instead, Carnap chose another topic at the boundary between physics and philosophy for his dissertation. A revised version of it was published, soon thereafter, as "Der Raum" (1922).

Axiomatics was still involved in Carnap's new dissertation, although now in a different way. Geometry had long been presented axiomatically; but during the late nineteenth century, it had been recast in a more "formal" axiomatic way, culminating in Hilbert's well-known *Grundlagen der Geometrie* (1899). Similar approaches to other branches of mathematics, including arithmetic and analysis, had also gained prominence during this period, through works by Dedekind, Peano, Hilbert, and others.[3] Carnap's interest in axiomatics stemmed, directly or indirectly, from these mathematical sources. Indeed, later he often referred to Hilbert's *Grundlagen* in this connection, as well as to "Axiomatisches Denken" (1918), an article in which Hilbert reflects programmatically on the development of the axiomatic method in mathematics, mentioning physics along the way. An axiomatic approach to both mathematics and physics, as championed by Hilbert, was thus a central goal for Carnap from

[3] See Awodey and Reck (2002a) for an overview of these developments, with the focus on ensuing meta-logical and metamathematical questions that will become central below.

early on – in spite of the fact that both Frege and Russell had been critical of such an approach.[4]

After having finished "Der Raum," Carnap went on, in the early 1920s, to pursue two other research projects. The first was influenced by Russell's book *Our Knowledge of the External World* (1914a), which Carnap read with enthusiasm in 1921, and by his earlier studies of Kant and neo-Kantian views at the University of Jena. This project resulted in *Der logische Aufbau der Welt* (1928a). Carnap's second, but much less well-known project concerned pure logic and its applications to mathematics – my main topic. A tangible result of that second project was the publication of a small book, *Abriss der Logistik* (1929), one of the very first textbooks in modern logic. While published a year after Hilbert and Ackermann's more prominent *Grundzüge der theoretischen Logik* (1928), Carnap's *Abriss* – essentially finished in 1927, largely independent of *Grundzüge*, and circulated widely – also had significant influence, especially in Vienna, where Carnap taught at the time.

However, Carnap never presented *Abriss der Logistik* as a major intellectual achievement. It was not intended to be a substantive contribution to logic, but simply to make the tools of logic more widely accessible and to argue for their general usefulness. Moreover, *Abriss* was obviously quite derivative from *Principia Mathematica*, as Carnap himself was the first to emphasize. It had grown directly out of the notes he took when studying Whitehead and Russell's book in 1919–20, and Russell influenced it further through a correspondence Carnap initiated with him in 1921.[5] It thus appeared, at the time and later, that Carnap was merely popularizing (in *Abriss*) and applying (in *Aufbau*) Russell's new logic. Seen in this light, he was just one of a number of logicians who assimilated *Principia* in the late 1910s and early 1920s, a group that also included Hilbert and members of his school.[6]

However, this appearance is misleading in at least two respects. First, unlike almost all the other logicians in question, Carnap was

[4] For Frege's corresponding criticisms, see Reck and Awodey (2004, 135–166).

[5] In particular, Russell sent Carnap a hand-written 35-page summary of *Principia Mathematica*, in 1922, after Carnap had informed him of having trouble obtaining a copy of the book. For more on that summary, against the background of Carnap's general correspondence with Russell in the 1920s, see Reck (2004).

[6] For the assimilation of *Principia* in the Hilbert school, see Sieg (1999).

not only influenced by *Principia Mathematica* but also by Frege's earlier work. As we saw, he was influenced by the latter very directly – by attending Frege's classes, in 1910–14, and by studying his writings carefully, from 1920 on. Moreover, the particular way in which Frege had presented his logic in "Begriffsschrift I" and "Begriffsschrift II" made it natural for Carnap to adopt two stances that were unusual at the time: (i) from very early on, he worked with a higher-order logic based on simple types, as opposed to the ramified types of *Principia*; (ii) also from early on, Carnap used higher-order logic as an inferential framework, as opposed to a system for reconstructing all of mathematics within a corresponding theory of classes.[7] Influenced by Frege's critical discussion of Hilbert in "Logik in der Mathematik," there was also another difference: (iii) Carnap was more motivated than most to find a way of combining, and reconciling, the use of logic as a general inferential framework with a Hilbertian axiomatic approach.

This brings us to the second respect in which the appearance of Carnap as a mere popularizer and user of Russellian logic is misleading. He actually set out to provide, generally and systematically, a synthesis of Frege's and Russell's approach to logic, on the one hand, and Hilbert's approach to axiomatics, on the other.[8] This is what Carnap's second main research project from the 1920s was supposed to accomplish (thus aiming far beyond the more incidental, merely pedagogical role of *Abriss*). More concretely, this project was intended to result in a second research monograph (besides *Aufbau*), with the working title *Untersuchungen zur allgemeinen Axiomatik* (ASP RC 080-various). Carnap finished large parts of this monograph in manuscript form, which he then circulated among a group of logicians between 1928 and 1930. Many of Carnap's corresponding goals and themes were also mentioned in two little-known articles: "Eigentliche und uneigentliche Begriffe" (1927) and "Bericht über Untersuchungen zur allgemeinen Axiomatik" (1930c).

[7] Concerning the second point, Frege's presentation of logic in his lectures differs significantly from the presentations in his publications. In the latter, his logical system includes prominently a theory of classes, while in the former that theory is simply omitted; see Reck (2004) and Reck and Awodey (2004) for more.

[8] As he progressed, Carnap also tried to synthesize these two approaches with a third: constructivism, as championed by Kronecker, Brouwer, etc. I put this aspect aside in the present chapter since I take it to be less central for Carnap's work in the 1920s. See Bonk and Mosterin (2000) on this topic.

II. TOWARDS GENERAL AXIOMATICS

From Euclid's geometry on, the axiomatic method has been used for a number of different purposes. Traditionally, axiomatics was seen as a method for organizing the concepts and propositions of a science, such as geometry, in order to increase their clarity and certainty. While such goals are sometimes still appealed to in modern applications, they have become less central in the transformed axiomatics promoted by Hilbert and others. What has become crucial instead is the systematic investigation, by increasingly abstract and formal means, of three logical properties of an axiomatic system: (a) the independence of its axioms; (b) their consistency; and (c) their completeness.

In Hilbert's *Grundlagen*, the first of these properties is made especially prominent, largely as a response to nineteenth-century insights into the independence of Euclid's famous Parallel Postulate. Hilbert also spends considerable time establishing consistency results for his geometric axioms, more precisely relative consistency theorems (obtained by semantic means), as they are closely related to independence results in their method of proof. The issue of completeness comes up as well, but it is left unclear and unexplored in Hilbert's early writings – in spite of the fact that in Dedekind's earlier work on the natural numbers relevant results concerning what has come to be called "categoricity" had already been established. Indeed, the precise relation between "completeness" and "categoricity," or even the fact that they can be distinguished conceptually, was one of the issues left in need of clarification. Further progress in this connection was made in the early 1900s, in publications by E. V. Huntington, O. Veblen, and other "Postulate Theorists."[9] After that, it took until the 1920s for more systematic investigations to be attempted.

Research done by Hilbert and his school during the 1920s is known primarily for its sharp focus on consistency questions, now with the goal of obtaining absolute consistency proofs (by syntactic means), especially for arithmetic and analysis. While the issue of the completeness of axiomatic systems was not entirely ignored in the Hilbert school, it was another mathematician and logician who addressed it more fully and explicitly at the time: Abraham Fraenkel.

[9] See Scanlan (1991) and Awodey and Reck (2002a) for further discussion.

In the first edition of Fraenkel's *Einleitung in die Mengenlehre*, published in 1919, completeness does not yet play a prominent role; but in the second and revised edition (Fraenkel, 1923), there is a long section on "The Axiomatic Method" containing a detailed discussion. This discussion made an immediate and strong impression on Carnap, and he was soon exchanging ideas with Fraenkel about the topic, both in correspondence and in person.

What gradually became clear during this period, through Fraenkel's and subsequent work, was that several related notions of completeness should be clearly distinguished and their relationships then further investigated. A first important distinction is between the completeness of deductive systems, on the one hand, and the completeness of axiom systems for particular parts of mathematics, on the other. An example of the former is the completeness of (various deductive systems for) sentential and first-order logic, as established by Post (1921) and Gödel (1929), respectively, which was brought into sharper focus by Hilbert and his school during the 1920s in connection with the issue of "decidability."[10] The latter notion of completeness, concerning axiom systems for geometry, for the natural numbers, the real numbers, and so on turns out to be in need of additional distinctions and sub-division. And the core question in Carnap's exchanges with Fraenkel was precisely what exact form such sub-division should take.

Carnap first published ideas related to these exchanges in his article "Eigentliche und uneigentliche Begriffe" (1927); Fraenkel did so in the third, again revised, and significantly expanded edition of his *Einleitung* (1928). The two authors agreed on the need to distinguish between three notions of completeness for systems of axioms. Fraenkel now formulates these three notions as follows:

> [T]he completeness of a system of axioms demands that the axioms encompass and govern the entire theory based on them in such a way that every question that belongs to and can be formulated in terms of the basic notions of the theory can be answered, one way or the other, in terms of deductive inferences from the axioms. Having this property would mean that one couldn't add any new axioms to the given system (without adding to the basic notions) so that the system was "complete" in that sense; since every relevant proposition that was not in contradiction with the system of

[10] See Zach (1999) for a discussion of early work in the Hilbert School on this topic.

axioms would already be a consequence and, thus, not independent, i.e., not an "axiom" . . .

Closely related to this first sense of completeness, but by far not as far reaching and easier to assess, is the following idea: . . . In general, a number of propositions that are inconsistent with each other and that can, thus, not be provable consequences of the same system of axioms can nevertheless be compatible with that system individually. Such a system of axioms leaves open whether certain relevant questions are to be answered positively or negatively; and it does so not just in the sense of deducibility by current or future mathematical means, but in an absolute sense (representable by independence proofs). A system of axioms of that kind is, then, with good reason, to be called incomplete . . .

Quite different, finally, is another sense of completeness, probably characterized explicitly for the first time by Veblen . . . According to it a system of axioms is to be called complete – also "categorical" (Veblen) or "monomorphic" (Feigl–Carnap) – if it determines the mathematical objects falling under it uniquely in the formal sense, i.e., such that between any two realizations one can always effect a transition by means of a 1–1 and isomorphic correlation. (Fraenkel, 1928, 347–349; my translation)

As the reference at the end of this passage indicates, Fraenkel saw himself as having benefited from his exchanges with Carnap. Other references make clear that with respect to Fraenkel's first notion of completeness, which both he and Carnap saw as closely connected with the notion of "decidability [*Entscheidungsdefinitheit*]," they felt indebted to Hilbert and his students, especially Heinrich Behmann and Hermann Weyl.[11]

From our present point of view, Fraenkel's three notions of completeness can be characterized, more briefly and in updated terminology, as follows:[12]

(1) A system of axioms S is *deductively complete* if and only if for every proposition P in the relevant language either P or not-P is deducible from S.

(2) A system of axioms S is *semantically complete* if and only if there is no proposition P in the relevant language such that both S together with P and S together with not-P are satisfiable, i.e., have a model.

(3) A system of axioms S is *categorical* (or *monomorphic*, as opposed to *polymorphic*) if and only if all models of S are isomorphic.

The main question then raised by Fraenkel, and seized upon by Carnap, is how these three notions are related. In the third edition of his book Fraenkel makes some general suggestions in this connection, but it is hard for him to be more conclusive. The reason, in hindsight, is that a precise answer requires the specification of a definite systematic background theory, and Fraenkel did not have such a theory at his disposal.

It is exactly at this point that Carnap, attempting to make further progress, is able to utilize what he learned earlier from Russell and, especially, from Frege. He proposes to reformulate Fraenkel's question within the framework of higher-order logic, specifically a system of higher-order logic with simple types understood purely inferentially – precisely as Carnap had encountered it in Frege's logic classes and as spelled out, subsequently, in *Abriss*. As noted above, neither Frege nor Russell had used their systems of higher-order logic for similar purposes, since both were fundamentally critical of the axiomatic method. Those interested in general axiomatics, like Hilbert, Behmann, Weyl, and Fraenkel, had also not yet made this synthesizing step, at least not systematically and in print.[13] Carnap, by contrast, was ideally situated to take this step, not only because of his close familiarity with Frege's and Russell's ideas and his interest in Hilbert's axiomatics, but also because of his active exchanges with Fraenkel.

Moreover, Carnap had further motivations for pursuing such a project stemming from his more general philosophical goals. First, some of the central ideas and methods of the *Allgemeine Axiomatik* project, such as the use of higher-order logic with simple types, are also present in the *Aufbau* project. Also, while the main focus in *Aufbau* is on empirical concepts, not on the concepts of logic and pure mathematics, axiomatically introduced concepts are not only important in pure mathematics, but have many fruitful applications in the empirical sciences as well, especially in the

[11] For Behmann's contributions to logic and metamathematics, see Mancosu (1999). For more on Carnap's and Fraenkel's indebtedness to both Behmann and Weyl, see Reck (2004).

[12] The terminology of "deductive completeness," "semantic completeness," and "categoricity" as used in the following definitions is not entirely standard. For further discussion, including some equivalent and historically significant variants, see Awodey and Reck (2002a).

[13] Alfred Tarski, who was doing independent work along related lines, is an exception; I will say more about him below.

axiomatic development of theories in physics. Second, from early on in his career Carnap was interested in explicating the notion of mathematical truth. Following Frege and Russell, he saw himself as a logicist, thus as defending the claim that mathematical truth and logical truth are fundamentally the same. The axiomatic method, as used by Dedekind, Peano, Hilbert, and others, seemed to provide another important approach to this issue; and in this respect as well Carnap's work on general axiomatics promised a way of combining the Fregean and the Hilbertian approaches. I will briefly return to these two additional topics later.

III. CARNAP'S APPROACH AND ITS LIMITATIONS

The part of Carnap's *Untersuchungen* that was worked out most fully by him, and then circulated among logicians between 1928 and 1930, is its part I. It begins with the following programmatic statement:

> In the course of recent investigations into general properties of axiomatic systems such as: completeness, monomorphism (categoricity), decidability, consistency, etc., and into the problem of determining criteria for and the mutual relations between these properties, one thing has become increasingly clear: that the main difficulty with respect to these problems lies in the insufficient precision of the concepts used. The most important requirement for a fruitful treatment of them is: on the one hand, to establish explicitly the logical basis to be used in each case, as is usually not done with enough precision; and on the other hand, to give precise definitions for the concepts used on that basis. In what follows, my aim will be to satisfy those two requirements and, subsequently, to establish the fruitfulness of the established foundation by deriving a number of theorems of general axiomatics. (Carnap, 2000, 59: my translation)

Here we can already see how Carnap's project was meant to go beyond Fraenkel's: Fraenkel had not "established explicitly the logical basis to be used"; he had not given "precise definitions for the concepts used on that basis"; and he had not "derived a number of theorems of general axiomatics" (at least not the theorems Carnap had in mind). At the same time, Carnap's list of "completeness, monomorphism (categoricity), and decidability" conforms exactly to Fraenkel's threefold distinction (arranged in a different order). For the first notion,

called "semantic completeness" above, Carnap also uses the term "non-forkability" (in the sense that a system S which is semantically complete is not "forkable" at any proposition P, i.e., does not "branch" in the sense specified in Definition (2)).

As already indicated, Carnap uses higher-order logic with simple types as the "logical basis" for his investigation. I will say more about the main "definitions of the concepts used on this basis" shortly. But to understand Carnap's goals, it is most helpful to go straight to the main "theorems of general axiomatics" he intended to establish. There are three core theorems which, from a contemporary point of view, would be formulated as follows:

THEOREM 1: An axiomatic system S is consistent (no contradiction is deducible from it) if and only if it is satisfiable, i.e., has a model.

THEOREM 2: An axiomatic system S is semantically complete (non-forkable) if and only if it is categorical (monomorphic).

THEOREM 3: An axiomatic system S is deductively complete if and only if it is semantically complete (non-forkable).[14]

Theorems 2 and 3 together would, if true, establish that all three notions of completeness distinguished by Fraenkel and Carnap are equivalent. Also, Theorem 1 (used by Carnap in his attempt to establish Theorem 3) may remind us of Gödel's later completeness theorem for first-order logic (Gödel 1929, 1930). It is important to keep in mind, however, that Carnap is working in higher-order logic, not in first-order logic. But then a red flag should go up immediately, since we now know that in that broader context Theorem 1 (understood in a contemporary sense) is not correct, since the "only if" part fails; likewise for Theorem 3.[15] (Theorem 2 is an interestingly different case, as we will see later.)

[14] Theorem 1 corresponds to Carnap's "Satz 2.4.9" (Carnap, 2000, 100), Theorem 2 to "Satz 3.4.10" (Carnap, 2000, 138), and Theorem 3 to "Satz 3.6.1" (Carnap, 2000, 144).

[15] Let PA be the higher-order Dedekind–Peano axioms (assumed to be consistent); let G be the sentence shown to be true but not provable from PA in Gödel's Incompleteness Theorem. Then PA together with $\sim G$ is consistent but not satisfiable. This shows that the "only if" part of Theorem 1 fails. As neither G nor $\sim G$ is provable, PA is not deductively complete; but it is semantically complete, because categorical. That shows that the "only if" part of Theorem 3 fails. For more background, see Awodey and Reck (2002a).

To assume, as Carnap obviously did, that all three "theorems" are capable of being established may look like an elementary blunder from our present point of view. But we need to keep in mind that we are looking at these issues with hindsight, from a perspective that has benefited from subsequent developments. Indeed, Gödel's famous Incompleteness Theorems, which show most directly that Theorems 1 and 3 are false (if understood in a contemporary sense), came as a big surprise to many when they were first announced, in 1930, and published, in 1931. In this sense, Carnap's misguided confidence in being able to establish his theorems may be compared to Hilbert's parallel confidence, repeatedly expressed by him in the 1920s, in being able to establish the consistency of arithmetic, analysis, and perhaps even set theory by "elementary means," a confidence also shattered by Gödel's results. Note also that, had Theorems 2 and 3 turned out to be true, this would have provided a clear and direct answer to Fraenkel's question about the relationship of his three notions of completeness. Yet something – indeed, several things – went wrong in Carnap's approach, and we now need to identify the main sources of the problems.

The most basic problem with Carnap's approach is that, despite his stated intention to give precise and workable definitions for his main concepts within an explicitly specified logical framework, the definitions he provides are not adequate for his own purposes. The core difficulty is that there is an ambiguity in his definition of the notion of deducibility, or of logical consequence more generally. Put briefly, Carnap works with the following notion (a descendant of Russell's notion of "formal implication"):

DEFINITION: The proposition $Q(t_1, \ldots, t_n)$ is a *logical consequence* of the proposition $P(t_1, \ldots, t_n)$ if and only if $\forall x_1 \ldots \forall x_n(P(x_1, \ldots, x_n) \supset Q(x_1, \ldots, x_n))$ holds.[16]

As an illustration, consider the case where $P(t_1, \ldots, t_n)$ is the conjunction of the Dedekind–Peano Axioms, $Q(t_1, \ldots, t_n)$ is some sentence of arithmetic, $t_1, \ldots, t_n$ are the basic constants used (here zero and successor), and everything else is defined in terms of them. The crucial question now is what "holding" is supposed to

[16] Quantifying out the constants $t_1, \ldots, t_n$ has an effect similar to the now standard idea of varying the interpretation for all the non-logical symbols in the language.

mean – a point left deliberately vague and indeterminate in our formulation of the definition. If we assume that it means "being deducible in the given formal system," then what we have, in effect, is the contemporary notion of syntactic consequence within higher-order logic. If we assume that "holding" means something like "being true" (in the "universal domain" assumed by Carnap, following Frege and Russell), then what we have is close to the contemporary notion of higher-order semantic consequence.

Let us call the two notions just distinguished "syntactic consequence" and "semantic consequence." In principle, it is possible to adopt either one. But which of them is Carnap working with, particularly when he talks about "deducibility"? From a contemporary point of view, one would expect him to work with syntactic consequence, especially since that seems to be the notion built into deductive completeness as used in Theorem 3. Recall also Fraenkel's informal characterization of deductive completeness in *Einleitung* (1928), as quoted above, which Carnap seems to want to explicate. Similarly, one would expect syntactic consequence to be built into Carnap's notion of consistency as occurring in Theorem 1. Overall, however, Carnap leans more towards semantic notions in *Allgemeine Axiomatik*, which points in the direction of semantic consequence; and in so far as this is the case, his explications of Fraenkel's distinctions are not adequate, especially that of deductive completeness. But most importantly, Carnap simply does not seem to be clear about the difference between syntactic and semantic consequence, both of which he can be read as invoking, at different points in his discussion, as if they were equivalent.[17] In other words, he is implicitly working with an inchoate amalgam of the two notions, and this is directly affecting his understanding of Theorems 1 and 3.

I have focused on Carnap's deficient understanding of the notion of deducibility, or of logical consequence more generally, which affects his treatment of deductive completeness, as well as his treatment of consistency. Beyond that, the notions of "model," "satisfiability," and "isomorphism," as built into his definitions of semantic completeness and categoricity, are also not treated in the now standard

[17] See here, for example, Carnap (2000, 92–93), where he moves freely back and forth between the relation of logical consequence for P and Q as simply "holding [gelten]" and as "being provable."

way in *Allgemeine Axiomatik*. But this aspect is less consequential, and I will not go into the details here. Sufficient for present purposes is to note the following general point: because of the ambiguity in his core notions, Carnap's approach and his main theorems are problematic, especially Theorems 1 and 3. From a contemporary point of view they, too, turn out to be ambiguous (involving either syntactic or semantic consequence). Moreover, if one removes the ambiguity, then the two theorems either come out true but trivial, or false and refuted by Gödel's Incompleteness Theorems.[18]

Besides the specific problems pointed out so far, there is also a more general, though not unrelated, problem. Carnap tried to stay within a general Fregean and Russellian "universalist" approach to logic: he uses a single formal system, formulated in a fixed, all-encompassing background language, as the framework in which all logical reasoning is to take place. From within his framework he then tries to distinguish several notions of completeness, to define consistency, and so on.[19] But what these notions call for, from our point of view, is the distinction between object-language and meta-language – between statements within the object-language in which the axiom system is formulated and statements about this object-language from a metatheoretic standpoint. This distinction, as we now know, allows for clear definitions of both syntactic consequence and semantic consequence as precise metatheoretic notions; similarly for the other notions at issue. Carnap did not make such a distinction, and this may be seen as the deeper reason for the failure of his project.

This seems to be, in fact, exactly the conclusion to which Carnap himself would soon be led. But that happened only after showing his manuscript for part I to several logicians, including Fraenkel and

[18] If we work with semantic consequence throughout, then Theorems 1 and 3 are true but trivial. (This is the case not only if we use "universalist" semantic consequence along Carnap's lines, but also semantic consequence in the now standard "model-theoretic" sense.) If we work with syntactic consequence, then the "only if" parts of Theorems 1 and 3 are refuted by Gödel's Incompleteness Theorems; compare fn. 15.

[19] For instance, Carnap defines consistency for a (finite) system of axioms not metatheoretically, but as follows (notation updated): Suppose that $P(t_1, \ldots, t_n)$ is the conjunction of the given axioms. Then the axiom system is called *consistent* if and only if $\neg\exists Q\forall x_1 \ldots \forall x_n(P(x_1, \ldots, x_n) \supset (Q(x_1, \ldots, x_n) \wedge \neg Q(x_1, \ldots, x_n)))$ "holds" (same ambiguity as above); see Carnap (2000, 97).

Gödel, starting in 1928. He also presented the corresponding material publicly both in Vienna, in talks and in classes that same year, and at a conference in Prague, the "First Conference on Epistemology of the Exact Sciences," in 1929. The immediate responses to these presentations by Carnap were lively, especially at the Prague conference. As he wrote in a corresponding diary entry shortly thereafter:

> My lecture: Investigations in General Axiomatics; just a brief summary. But the proof is requested, and acknowledged. Though it was late, a lively discussion on the basic issues afterwards; von Neumann, Zermelo, Hahn, Fraenkel said that a final judgment will only be possible when the complete proof (especially of Theorem 2 and 3 above) is available. Amazing interest in my Investigations. (Quoted in Awodey and Carus, 2001, 162)

The list of logicians mentioned here is impressive: von Neumann, Zermelo, Hahn, and Fraenkel. Apparently none of them saw right away what had gone wrong with Carnap's proof, or with his approach more generally. While this is probably due in part to the fact that Carnap only presented "a brief summary," the reaction also indicates that, at the time, the issues were not well understood more generally. In this sense, too, Carnap was presenting material at the cutting edge of research in logic and metamathematics.

The turning point for Carnap's project occurred in 1930. Two events were crucial. First, early in 1930 Alfred Tarski visited Vienna, and the two had several conversations, including on the topic at hand. After one of these conversations, Carnap wrote the following revealing remark in his diary:

> Tarski visits me . . . talked about my *Axiomatik*. It seems correct, but certain concepts don't capture what is intended; they must be defined metamathematically rather than mathematically. (Quoted in Awodey and Carus, 2001, 163)

Apparently Tarski – who already had made independent progress on how to frame, and then pursue, general questions in logic from a metatheoretic standpoint (although this work was only starting to become known outside Poland) – pointed out exactly the general problem with Carnap's approach as described above.

The second crucial event involved Kurt Gödel. During the second half of the 1920s Gödel was a research student at the University of Vienna, and Carnap had frequent conversations him, especially on logical matters. Gödel also attended Carnap's talks and classes in Vienna, in 1928, in which material from *Allgemeine Axiomatik* was presented, and he was one of the people who received a copy of Carnap's manuscript. Thus, the two had direct and prolonged contact in connection with exactly the issues under discussion in this chapter. At least partly influenced by that contact (partly also, as is well known, by work in the Hilbert school), Gödel then came up with his celebrated results: the Completeness Theorem for first-order logic and the Incompleteness Theorems for arithmetic and higher-order logic. The latter were first publicly announced at a conference in Königsberg, in the fall of 1930; but there is evidence that Gödel had told Carnap about them already earlier, during a conversation in August of that year. And while Carnap had problems following the details of the corresponding proofs initially, he recognized the importance of Gödel's results very quickly.[20]

When Gödel announced his Incompleteness Theorems for the first time publicly, at the Königsberg conference, it was in connection with his Completeness Theorem. After reporting on a proof of the latter in detail, he remarked:

I would furthermore like to call attention to an application that can be made of what has been proved [the Completeness Theorem] to the general theory of axiom systems. It concerns the concepts "decidable [*entscheidungsdefinit*]" and "monomorphic" . . . One would suspect that there is a close connection between these two concepts, yet up to now such a connection has eluded general formulation . . . In view of the developments presented here it can now be shown that, for a special class of axiom systems, namely those whose axioms can be expressed in the restricted functional calculus [i.e., first-order logic], decidability [*Entscheidungsdefinitheit*] always follows from monomorphism . . . If the completeness theorem could also be proved for the higher parts of logic (the extended functional calculus) [including the logic of *Principia Mathematica* and Carnap's simple type theory], then it would be shown in complete generality that decidability follows from monomorphism; and since we know, for example, that the Peano axiom system is monomorphic, from that the solvability of every problem of arithmetic and analysis in *Principia Mathematica* would follow.

[20] For more on the interactions between Carnap and Gödel in this connection, see Awodey and Carus (2001), Goldfarb (2003), and Goldfarb (2005).

Such an extension of the completeness theorem is, however, impossible, as I have recently proved; that is, there are mathematical problems which, though they can be expressed in *Principia Mathematica*, cannot be solved by the logical devices of *Principia Mathematica*. (Quoted in Goldfarb, 2005, 190–192; translation slightly amended)

Several details in this passage are significant in our context, since they show that Gödel's way of looking at the issue was very much influenced by Carnap's *Allgemeine Axiomatik* project. Note especially the terminology of "decidable" and "monomophic," as well as the question about the relation between these two notions. Note also Gödel's remark that "one would suspect that there is a close connection between these two notions." It seems that he thought his audience would agree that the latter was a natural suspicion – which, of course, made his "recent proof" that it is false more significant.

Thus, it was Tarski who first convinced Carnap, in early 1930, that his general framework was inadequate; but it was Gödel who directly showed him, later in 1930, that, even if the approach could be formulated adequately, several of Carnap's main theorems could not be salvaged. As a result of both of these blows, Carnap did not pursue part I of his project further after 1930 – the manuscript for it disappeared in a drawer, to be rediscovered and published only seven decades later. Indeed, Carnap became so convinced of the futility of this project that later, in his "Intellectual Autobiography" (1963a), he didn't even mention it. The only remaining traces were the corresponding remarks in "Eigentliche und uneigentliche Begriffe" (1927) and the summary in "Bericht über Untersuchungen zur allgemeinen Axiomatik" (1930c). It seems that the latter article had already gone into press when Carnap abandoned the project, so that its publication could no longer be prevented.

IV. AFTERMATH AND CONTINUING SIGNIFICANCE

At this point, the question arises why we should pay attention to *Allgemeine Axiomatik* today. Wasn't Carnap right to ignore it? It seems to me that, despite its general failure, the project is worthy of contemporary attention. Carnap was addressing important issues, issues that remain of interest for several reasons. Some of these involve technical, mathematical questions in general axiomatics; others have to do with Carnap himself, especially with a better understanding of the development of his views; yet others concern the history and philosophy of logic and metamathematics more generally.

A first observation to make in this connection is that from the perspective of a general investigation into the strengths and weaknesses of the axiomatic method the issues addressed by Carnap are undoubtedly central. More specifically, while in the work of Hilbert and his school the focus was on the notions of independence and consistency, the notion of completeness for systems of axioms is equally important – as acknowledged by everyone in the 1920s, including Hilbert himself. Moreover, this notion becomes particularly interesting within the higher-order logical framework adopted by Carnap. If we restrict ourselves to first-order logic, few mathematical theories are complete in any of the senses distinguished above. By contrast, in higher-order logic the axiomatic theories of the natural and real numbers, as well as of Euclidian geometry, are all semantically complete and categorical. For these reasons, higher-order logic seems to be the most natural framework for investigating notions of completeness.

As we have seen, Carnap thought that his three notions of completeness – "decidability" (deductive completeness), "nonforkability" (semantic completeness), and "monomorphism" (categoricity) – are all equivalent. This is wrong on several counts in the context of first-order logic. In the context of higher-order logic, the equivalence of deductive completeness and semantic completeness – asserted in Theorem 3 – is also false, as pointed out by Gödel at the Königsberg conference. To be more precise, while it is true that deductive completeness implies semantic completeness, Gödel's results refute the converse implication. However, what about the alleged equivalence of semantic completeness and categoricity in higher-order logic – Carnap's Theorem 2? This equivalence again involves two directions. One, from categoricity to semantic completeness, is correct and relatively easy to establish (not only for first-order logic, but also for higher-order logic). The other direction is much harder, and still not completely clarified. Here the question remains: Is it the case that an axiomatic system that is semantically complete is thereby also categorical? Carnap thought that the answer was positive, a claim we might therefore call "Carnap's Conjecture."[21]

Not only did Carnap think that this conjecture is true, he believed he had found a proof. Unfortunately, while neither the general inadequacy of his approach pointed out by Tarski nor the more specific results by Gödel immediately refute Carnap's work on this point, there is an additional weakness in his treatment not mentioned so far. Carnap made a subtle implicit assumption in his attempted proof that was later shown not to be true in general. With this assumption added the proof is correct, but it does not establish Carnap's Conjecture, only a partial, qualified result.[22] This leads to a new question: Might there not be some other proof of the conjecture, one not relying on any such additional assumption? As far as I am aware, this question is still unsettled, and so Carnap's Conjecture, in full generality, remains an open question.[23] What we have here is a natural and central question in general axiomatics still awaiting an answer, one to which a reconsideration of Carnap's 1920s project directly leads us and which may prove more tractable now.

I have concentrated so far on part I of *Allgemeine Axiomatik*, mostly because we only have this part available in print (as Carnap, 2000); but a few remarks about part II can illustrate further the remaining significance of Carnap's work. Here Carnap intended to address a number of further questions connected with his three notions of completeness, specifically questions involving "extremal axioms." An example of such an axiom is Hilbert's "Axiom of Completeness" in his axiomatization of Euclidian geometry, which can be considered a "maximality axiom"; another example is Dedekind's and Peano's induction axiom, forming part of their respective axiomatizations of arithmetic, which constitutes a "minimality axiom."

21 It is implicitly assumed here, as it was during Carnap's time, that axiom systems have to be finite. Without that assumption the conjecture can be shown to be false; see Awodey and Reck (2002b, 83).

22 Carnap assumed that every model of a higher-order theory is definable. This is made explicit in the correct, but restricted, version of the result published, a few years later, in Lindenbaum and Tarski (1935). For more on this issue, see Awodey and Reck (2002a) and, especially, Awodey and Reck (2002b).

23 The conjecture is known to be true in some special cases, e.g., when working with simple type theory with no non-logical constants ("pure higher-order logic"); see Awodey and Carus (2001, 160–161).

As Carnap observed, both of these axioms lead to categorical theories. This suggests the question how and to what extent this phenomenon generalizes, i.e., whether other "extremal axioms" can be found that have the same effect and, if so, why they have that effect. Once again, this amounts to a natural and central question in general axiomatics, with many implications and subquestions.

While Carnap himself never reached a full answer to this question, he addressed parts of it successfully. He also rescued several of the results obtained here from the rubble of *Allgemeine Axiomatik* and published them a few years later in "Über Extremalaxiome" (1936), written in collaboration with Friedrich Bachmann. Like his other publications in logic from the period, this article did not draw a lot of attention, not least because the questions and results in it were now presented out of context; they thus lacked the support of the more general project within which Carnap studied them. Nevertheless, they led to a few subsequent investigations, e.g. in Fraenkel and Bar-Hillel's *Foundations of Set Theory* (1956). Also, given the recent broadening and branching out of logic, including a revival of higher-order logic (in computer science and category theory, among others), some of Carnap's results and conjectures in this connection too might prove fruitful for future research.[24]

As mentioned above, for Carnap the *Allgemeine Axiomatik* project also had broader philosophical significance, especially in two respects. First, he saw it as connected with general questions about the applicability of mathematical concepts to the empirical world, as investigated contemporaneously in *Der logische Aufbau der Welt*. This connection, in fact, was the main topic of the article "Eigentliche und uneigentliche Begriffe" (1927), in which Carnap argued that not only explicitly defined concepts, as treated in the *Aufbau*, but also concepts introduced by "complete" systems of axioms were of special importance for science, and thus in need of further clarification. Although this argument has not yet found much attention in the secondary literature, exploring it further might again prove fruitful in the future, now in connection with Carnap's views about empirical knowledge.[25]

[24] For more on this general issue, see Awodey and Reck (2002b).

[25] See Awodey and Carus (2001), in which this issue is emphasized. For a different perspective, and a rare earlier discussion of Carnap (1927), compare Goldfarb (1996).

Second and independently, Carnap also saw a direct connection between questions about completeness and the notion of mathematical truth, and thus the philosophy of mathematics. Here the basic idea – already implicit in earlier works on geometry by Hilbert and on the natural numbers by Dedekind and Peano – is this: If a mathematical theory can be based on a "complete" set of axioms, then the notion of truth in that area is captured fully in terms of the "logical consequences" of these axioms. After having learned, from Tarski and Gödel, about the problems with his treatment of completeness and logical consequence in *Allgemeine Axiomatik*, Carnap could, of course, no longer simply uphold this basic idea; at the very least, it needed to be modified and clarified. But that leads to the following question: Which modifications, if any, are possible? In other words, is there some less problematic variant of this approach that still provides us with an axiomatic, and broadly logicist, notion of mathematical truth?

The latter question remained very much a concern for Carnap after giving up his *Allgemeine Axiomatik* project in 1930. In response to Tarski's ideas, he now fully embraced the object- versus metalanguage distinction. Indeed, in his later reflections he characterized one of the main goals for his work in the early 1930s as follows:

> One of my aims [at that point] was to make the metalanguage more precise, so that an exact conceptual system for metalogic could be constructed in it. Whereas Hilbert intended his metamathematics only for the specific purpose of proving the consistency of a mathematical system formulated in the object language, I aimed at the construction of a general theory of linguistic forms. (Carnap, 1963a, 53)

Carnap then attempted, on such a basis, to provide a post-Gödelian characterization of mathematical truth, one that takes full account of Gödel's Incompleteness Theorems. Indeed, the pursuit of this goal is a theme that connects many of Carnap's publications from the 1930s on: from *Die Logische Syntax der Sprache* (1934c) to "Die Antinomien und die Unvollständigkeit der Mathematik" (1934a) and "Ein Gültigkeitskriterium für die Sätze der klassischen Mathematik" (1935a), and even to later writings such as *Introduction to Semantics* (1942) and *Meaning and Necessity* (1947).[26] In these

[26] Both Carnap (1934a) and Carnap (1935a) were later worked into the augmented English edition of *Logische Syntax* (Carnap, 1934c/1937), as indicated on p. xi of its preface.

works, Carnap tried out a variety of ideas for how to capture the notion of mathematical truth along the general lines indicated above: broadly syntactic ideas, semantic ideas, infinitary logic, and even modal logic. Unfortunately, he never arrived at a satisfactory solution.

A full understanding of Carnap's views and their development will have to incorporate an account of his various attempts in this connection. This issue goes far beyond the bounds of the present chapter,[27] but let me make one further observation concerning it. While it is not inappropriate to think of these Carnapian investigations in the philosophy of mathematics as his response to Gödel's Incompleteness Theorems, a reconsideration of Carnap's earlier *Allgemeine Axiomatik* project reveals that their roots go back considerably further. They go deep into the 1920s, partly even the 1910s; and they consist of Carnap's early attempts at combining a Fregean and a Hilbertian point of view in logic and axiomatics. Thus, Gödel's results were not really the starting point in this connection. Rather, Gödel's results had special significance for Carnap precisely because of his own earlier work on logic and axiomatics. Moreover, that earlier work by Carnap provided a significant part of the framework within which Gödel himself proceeded.

This brings me to a final observation about the significance of the *Allgemeine Axiomatik* project, now for the history and philosophy of logic more generally. If, as I have argued, Carnap's project, despite its flaws, was at the cutting edge of research in logic and early metamathematics, then new light is shed on their development. It is often acknowledged that the 1920s was one of the most active and fertile decades in the rise of modern logic. But this is typically, and sometimes exclusively, acknowledged in connection with Hilbert's influential work (his quest for consistency proofs, his confrontation with Brouwer, and so on); and as a consequence, Gödel's theorems are interpreted as fitting into that development. While such an interpretation does shed significant light on the situation, paying attention to Carnap's *Allgemeine Axiomatik* project allows us to understand another, partly independent development also leading to Gödel. As a result, a fuller picture of the motivations for his celebrated results emerges. Gödel's work now appears as a node in a whole web of

[27] See Steve Awodey's contribution to this volume for a detailed discussion.

research during the 1920s and early 1930s, one that ties together not just Hilbert and Gödel, but also Frege, Russell, Fraenkel, and Tarski – with Carnap as a central mediating figure.[28]

[28] I am grateful to the editors, Michael Friedman and Richard Creath, for inviting me to contribute to this volume. Many thanks also to André Carus, William Demopoulos, Michael Friedman, and Paolo Mancosu for helpful comments on drafts of this chapter, as well as to Steve Awodey for collaborations that put me in a position to write it.