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STRUCTURES AND STRUCTURALISM IN CONTEMPORARY PHILOSOPHY OF MATHEMATICS

ABSTRACT. In recent philosophy of mathematics a variety of writers have presented "structuralist" views and arguments. There are, however, a number of substantive differences in what their proponents take "structuralism" to be. In this paper we make explicit these differences, as well as some underlying similarities and common roots. We thus identify systematically and in detail, several main variants of structuralism, including some not often recognized as such. As a result the relations between these variants, and between the respective problems they face, become manifest. Throughout our focus is on semantic and metaphysical issues, including what is or could be meant by "structure" in this connection.

1. INTRODUCTION

In recent philosophy of mathematics a variety of writers – including Geoffrey Hellman, Charles Parsons, Michael Resnik, Stewart Shapiro, and earlier Paul Benacerraf – have presented "structuralist" views and arguments.[1] As a result "structuralism", or a "structuralist approach", is increasingly recognized as one of the main positions in the philosophy of mathematics today. But what exactly *is* structuralism in this connection? Geoffrey Hellman's discussion starts with the following basic idea:

> [M]athematics is concerned principally with the investigation of structures of various types in complete abstraction from the nature of individual objects making up those structures. (Hellman 1989, vii)

Charles Parsons gives us this initial characterization:

> By the "structuralist view" of mathematical objects, I mean the view that reference to mathematical objects is always in the context of some background structure, and that the objects have no more to them than can be expressed in terms of the basic relations of the structure. (Parsons 1990, 303)

Such remarks suggest the following *intuitive theses*, or *guiding ideas*, at the core of structuralism: (1) that mathematics is primarily concerned with "the investigation of structures"; (2) that this involves an "abstraction from the nature of individual objects"; or even, (3) that mathematical objects

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"have no more to them than can be expressed in terms of the basic relations of the structure". What we want to do in this paper is to explore how one could and should understand theses such as these.

Our main motivation for such an exploration is the following: When one tries to extract from the current literature how structuralism is to be thought of more precisely – beyond suggestive, but vague characterizations such as those above – this turns out to be harder and more confusing than one might expect. The reason is that there are a number of *substantive differences* in what various authors take structuralism to be (as already hinted at in the distinction between (2) and (3)). Moreover, these differences are seldom acknowledged explicitly, much less discussed systematically and in detail.[2] In some presentations they are even blurred to such a degree that the nature of the view, or views, under discussion remains seriously ambiguous.[3] Our main goals is, then, to make explicit the differences between various structuralist approaches. In other words, we want to exhibit the *varieties of structuralism* in contemporary philosophy of mathematics.

At the same time, the emphasis of our discussion will be systematic rather than exegetical. That is to say, while keeping the current literature in mind, we will attempt to lay out a coherent *conceptual grid*, a grid into which the different variants of structuralism will fit and by means of which the relations between them will become clear. This grid will cover several ideas and views – by W. V. Quine, Bertrand Russell, and others – that are usually not called "structuralist" in the literature, an aspect that will make our discussion more inclusive than might be expected. The discussion will also have a historical dimension, in the sense that we will identify some of the historical roots of current structuralist views. On the other hand, we will restrict our attention to structuralist views in the *philosophy of mathematics*, not beyond it. As a consequence the paper will center around certain *semantic* and *metaphysical* questions concerning mathematics, including the question what is or could be meant by "structure" in this connection.

2. BACKGROUND AND TERMINOLOGY

It will clarify our later discussion if we remind ourselves first, briefly, of some basic mathematical and logical definitions, examples, and facts. For many readers these will be quite familiar; but we want to be explicit about them, especially since some of the details will matter later. We also want to introduce some terminology along the way.

Our first and main example will be *arithmetic*, i.e., the theory of the natural numbers $1, 2, 3, \ldots$. This is the central example in most discussions of structuralism in contemporary philosophy of mathematics. In contemporary mathematics it is standard to found arithmetic on its basic axioms: the *Peano Axioms*; or better: the *Dedekind–Peano Axioms* (as Peano acknowledged their origin in Dedekind's work). A common way to formulate them, and one appropriate for our purposes, is to use the language of 2nd-order logic and the two non-logical symbols '1', an individual constant, and s, a one-place function symbol ('s' for "successor function"). In this language we can state the following three axioms:

(A1) $\forall x[1 \neq s(x)]$,

(A2) $\forall x \forall y[(x \neq y) \rightarrow (s(x) \neq s(y))]$,

(A3) $\forall X[(X(1) \wedge \forall x(X(x) \rightarrow X(s(x)))) \rightarrow \forall x X(x)]$.

Let $PA_2(1, s)$, or more briefly PA_2, be the conjunction of these three axioms.[4]

A second example we will appeal to along the way, although in less detail, is *analysis*, i.e., the theory of the real numbers. Using 2nd-order logic and the non-logical symbols '0', '1', '+', and '·' we can again formulate certain corresponding axioms, namely the *axioms for a complete ordered field*. Let COF_2 be the conjunction of these axioms. As a third example, used mainly for contrast, we will occasionally look at *group theory*. Here we can restrict ourselves to 1st-order logic and the non-logical symbols '1' and '·'. Let G be the conjunction of the corresponding *axioms for groups*.[5]

From a mathematical point of view it is now interesting to consider PA_2 as a certain *condition*. That is to say, we can study various systems – each consisting of a set of objects S, a distinguished element e in the set, and a one-place function f on it – with respect to the question whether they satisfy this condition or not, i.e., whether the corresponding axioms all hold in them. Analogously for COF_2 and for G. So as to be able to talk concisely here, let us call the corresponding systems *relational systems*. Thus a relational system is some set with one or several constants, functions, and relations defined on it. Furthermore, let us call any relational system that satisfies PA_2 a *natural number system*. Analogously for *real number systems* and for *groups*.[6]

Here are a few basic facts concerning natural number systems: Assume that we are given some infinite set U (in the sense of Dedekind-infinite).[7] Then we can construct from it a natural number system with underlying set $S \subseteq U$, i.e., a relational system that satisfies PA_2. Actually, there is then not just one such system, but many different ones – infinitely many (all based on the same set S, and we can construct further ones by varying that set). At the same time, all such natural number systems are isomorphic. A similar situation holds for real number systems. One noteworthy difference is, of course, that here we have to start with a set of cardinality of the continuum. (Remember that we are working with 2nd-order axioms, in both cases.) Given one such set we can, again, construct infinitely many different real number systems; and all of them turn out to be isomorphic. In the case of G the situation is quite different. Here we do not need an infinite set to start with; and given some large enough set we can construct many groups that are non-isomorphic. In usual mathematical terminology: PA_2 and COF_2 are *categorical*, while G is not; and all of them are *satisfiable*, thus *relatively consistent*, given the existence of a large enough set.

Let us recast these facts in a more thoroughly set- and model-theoretic form. Assume, first, that we have standard ZFC set theory (where we deal merely with pure sets, not with additional urelements) as part of our background theory. In this case, we have a large supply of pure sets at our disposal, large enough for most mathematical purposes: all the sets whose existence ZFC allows us to prove. We can also think of relations and functions on these sets as pure sets themselves, namely as sets of ordered pairs, triples, etc. As a consequence we can work with *purely set-theoretic relational systems*, themselves defined as certain set-theoretic n-tuples. Next, assume that we treat our given arithmetic language explicitly as a formal language, i.e., as uninterpreted. That allows us to talk, in the technical (Tarskian, model-theoretic) sense, about various *interpretations* of that language, as well as about various relational systems being *models* of the axioms or not. (A model is a relational system considered relative to a given set of axioms.) In fact, we can go one step further: we can treat arithmetic language itself as consisting just of sets. If we do so, we can conceive of interpretations, models, etc. *entirely* in set-theoretic terms.

Given this setup the ZFC axioms allow us to prove a number of familiar results. Consider the triple consisting of the set of finite von Neumann ordinals ω, the distinguished set $\emptyset$ in it, and the successor function $s_N : x \mapsto x \cup \{x\}$ defined on it. This system is a natural number system. Similarly, the finite Zermelo ordinals ω', with distinguished element $\emptyset$ in it, and the alternative successor function $s_Z : x \mapsto \{x\}$ defined on it, form a natural number system. Also, clearly these two relational systems are not identical (in the sense that they consist of different sets), while they are isomorphic. Finally, starting with either one of them we can construct infinitely many additional natural number systems: by rearranging the elements of the systems in an appropriate way; by leaving out finite initial segments; by leaving out all the "odd numbers", all the "even numbers", or all the "prime numbers"; by adding new finite sets as initial segments; etc.

A main advantage of working with pure ZFC set theory is that within it everything is determined by explicit axioms and definitions. We can go slightly beyond it by adding some sets that are not pure. The usual, precise way to do so is by starting with a set, or even a proper class, of *urelements* and by then building the set-theoretic hierarchy on top of it. We then still have all the pure sets at our disposal, thus all the relational systems from above, but we can also build urelements into various new relational systems. In particular, we can form many further models of PA_2. To make this result more graphic: We can start with *any* object – a grain of sand, the Sears Tower, or the Moon – and build *it* into a model of PA_2. That is to say, any object whatsoever can be used as the base element of a natural number system, or as its 2nd element, its 17th element, etc. Finally, if we assume – a strong additional assumption, to be sure – that we have an infinite set of urelements to start with, we can build models of PA_2 whose underlying sets consist *entirely* of such urelements. Analogously for COF_2 and for G.

3. FROM MATHEMATICAL PRACTICE TO FORMALIST STRUCTURALISM

So much for general background and terminology. We now want to identify a certain structuralist approach that is shared by many contemporary mathematicians. As it concerns what mathematicians do, we will call it their *structuralist methodology*. This methodology motivates, explicitly or implicitly, many of the structuralist views in the philosophical literature, as we will see later.

Consider the entities most contemporary mathematicians simply assume in their everyday practice: the natural numbers, the integers, the rational, real, and complex numbers, various groups, rings, modules, etc., different geometric spaces, topological spaces, function spaces, and so forth. Mathematicians with a structuralist methodology stress the following two principles in connection with them: (i) What we usually do in mathematics (or, in any case, what we should do) is to study the *structural features* of such entities. In other words, we study them as structures, or insofar as they are structures. (ii) At the same time, it is (or should be) of *no* real concern in mathematics what the *intrinsic nature* of these entities is, beyond their structural features. Thus, all that matters about the natural numbers mathematically, however we think about them otherwise, is that they satisfy PA_2 (including, of course, what follows from that fact); all that matters about the real numbers is that they satisfy COF_2. Similarly, what matters is that various sets together with some given constants, functions, and relations defined on them form groups, rings, modules, etc. Put briefly, the proper business of mathematics is to study these and similar structural facts, and *nothing else*.

Looked at historically, this structuralist methodology is the result of several important innovations in 19th and early 20th century mathematics. To mention four of them, very briefly:[8] First, there is the rise of "abstract algebra", i.e., the development of group theory, ring theory, field theory, etc. as we know them today. Such theories involve a focus on certain general, abstract features, like those defined by the group axioms, that are shared by many different systems of objects. This leads naturally to a structuralist attitude with respect to the subject matter of these fields; actually, it is hard to see how else to think about them. Similarly for topology, functional analysis, etc. Second, even with respect to the older, more "concrete" parts of mathematics – arithmetic, the Calculus, the traditional study of geometry – we find the use of the formal, axiomatic method during this period, i.e., the formulation of axiom systems like PA_2, COF_2, and Hilbert's axioms for geometry. With respect to these parts of mathematics, too, a structuralist point of view is thus made possible, although it is not as much forced on us as in the case of algebra and topology.

Third, there is the introduction and progressive development of set theory in the 19th and early 20th century, leading to the formulation of the ZFC axioms. Set theory provides, then, a general framework in which all the other parts of mathematics can be unified and treated in the same way.[9] That is to say, in set theory one can construct various groups, rings, fields, geometric spaces, topological spaces, as well as models for PA_2, for COF_2, etc. (see Section 2); and one can study them all in the structuralist way described above. Fourth, such a structuralist approach to mathematics, within the framework provided by set theory, is then made canonical, at least for large parts of 20th century mathematics,[10] with the influential, encyclopedic work of *Bourbaki* and his followers. Consequently it is with the name of Bourbaki that "structuralism in mathematics" is most often associated in the minds of contemporary mathematicians.[11]

All of this concerns mathematical practice. From a philosophical point of view one now wants to go a step further and ask: How should we understand such a structuralist methodology in terms of its *philosophical implications*? As it stands this is a rather general and vague question, i.e., it needs further specification. The way many contemporary philosophers of mathematics (as well as philosophers of language and metaphysicians) specify it further is this: How are we supposed to think about *reference* and *truth* along these lines, e.g., in the case of arithmetic? And what follows about the *existence* and the *nature* of the natural numbers, as well as of other mathematical objects, even if the answer doesn't matter mathematically? Put more briefly, what are the *semantic* and *metaphysical* implications of a structuralist methodology?[12]

Adopting our structuralist methodology does not, in itself, answer these questions. This methodology is relatively *neutral* with respect to them. For many mathematical and scientific purposes such neutrality is probably an advantage. For philosophical purposes, on the other hand, especially those informing much of contemporary philosophy of mathematics, it is not really satisfying – since then the question becomes simply: Which semantic and metaphysical *additions* are most consonant, or at least consistent, with a structuralist methodology? As we will see, there are several *different, competing* ways of adding to a structuralist methodology in this sense. Let us briefly consider three of them which are rather "thin" or "formalist" in the rest of this section, before turning in more detail to several "thick" or "substantive" alternatives later.

A first and rather negative addition to our structuralist methodology, now less common than it used to be, is to simply reject all questions about the "real nature" of numbers, about "the referents" of numerical expressions, and about "mathematical truth". More particularly, one may suggest that the corresponding semantic and metaphysical questions are either *meaningless* or in some other way *misguided*; thus that they should be avoided not only in mathematics, but also in the philosophy of mathematics. However, in this case it seems fair to ask back what exactly is problematic about these questions, since on the surface they seem to be meaningful and interesting. Various answers to that question may be suggested in turn (e.g., along Carnapian or Wittgensteinian lines).

A second, still pretty negative or minimal, position is to fall back on some kind of formalism (understood in a narrow sense) at this point. That is to say, one can try to supplement our structuralist methodology with the following thesis: What we really deal with in mathematics, or at least in pure mathematics, are just *empty signs* in the end, i.e., signs used to play certain *formal games*, but not to be, as such, "about" anything.[13] In such a case our philosophical questions above are not exactly rejected, but given a deflationary or "thin" answer. But here, too, we can ask back: Is this not too radical a response, i.e., isn't there something "behind" the formalisms in mathematics? Also, what exactly is meant by "playing formal games" in this context? Such a response needs then again further elaboration and defense.

Third and perhaps most promisingly, one can try to add the following thesis to methodological structuralism: What mathematicians really study are not any objects and their properties, but certain general *inference relations* or *inference patterns*. After all, doesn't it seem that what we do in mathematics is primarily to study, in a systematic way, what can and what cannot be inferred from various kinds of basic principles, e.g., from PA_2, COF_2, or G? However, this proposal leads quickly to some new questions, including: What exactly does speaking of "inference relations" here involve; in particular, what are the *relata*: mere sentences (so that we are back to some kind of formalism?), propositions (leading us beyond formalism after all?), etc.? Or are we supposed to understand the nature of the inference relations in some radically different way?[14]

Note that with suggestions such as the three just listed we *have* responded to the semantic and metaphysical questions raised above, negative or "thin" as these responses may be. To be able to contrast this general kind of response with others later on, let us give it a name; let's call it *formalist structuralism* ("formal" now in a broader sense, as opposed to "substantive"). Formalist structuralism consists, thus, in endorsing a structuralist methodology for mathematics while responding to our semantic and metaphysical questions by either rejecting or deflating them, in one of the three ways mentioned.

Formalist structuralism is not the only philosophical position consistent with adopting a structuralist methodology in mathematics. In fact, one can admit that we need not be concerned about the deeper, real nature of the natural numbers when doing arithmetic, but argue – on additional, philosophical grounds – that they nevertheless *have* such a nature. One can even try to hold on to some kind of *platonism* or *realism* about the natural numbers in the sense of defending the thesis that they are special, particular abstract objects, to be thought of in an essentially non-structuralist way. However, such a move may now appear quite alien to, or at least curiously unconnected with, mathematical practice. And this may lead us to several *more substantive variants of structuralism.*[15]

4. RELATIVIST STRUCTURALISM

Formalist structuralism gives negative, minimal, or "thin" answers to the semantic and metaphysical questions central to much contemporary philosophy of mathematics. We now want to turn to one version of structuralism that offers more substantive, "thick" answers to them. We will call this version *relativist structuralism*, for reasons that will become clear shortly. With respect to characterizing it our reminders above, about some basic mathematical and logical facts, will be especially useful.

Let us start again by considering arithmetic, axiomatized in terms of PA_2, as our main example. Let us also assume, as earlier, that we use only the two non-logical symbols '1' and 's'. This means that the other

arithmetic symbols, '$<$', '$+$', '$\cdot$', also '2', '3', etc., have all been defined in terms of these two (using explicit and inductive definitions, as usual). If we now consider some ordinary arithmetic sentence p, e.g., '$2 + 3 = 5$' or '$\forall x \forall y \forall z[(x + y) + z = x + (y + z)]$', then there is a *translation* $p(1, s)$ of it that contains only '1' and 's'. And if we ask what p is "about", all we have to do is this: to ask what the symbols '1' and 's' in $p(1, s)$ refer to, as well as what the quantifiers in it (if there are any) range over. Finally, let us assume, at least initially, that some infinite set exists. Then we know that there are infinitely many models of PA_2.

Given these assumptions, what can we say – along structuralist lines, but not in the formalist sense – about the reference of '1' and s' and about the range of the quantifiers in $p(1, s)$? A relativist structuralist offers the following response: We simply *pick* one particular model M of PA_2, consisting of a domain S, a distinguished element e in S, and a successor function f on S (here M can be some model that is particularly convenient for the purposes at hand, but it doesn't have to be); and we *stipulate* that '1' refers to e, that 's' refers to f, and that the range of the quantifiers is S. At the same time, we note that we could also have picked *any other* model M' of PA_2. In that case '1' would have referred to the base element e' in its domain S', 's' to the successor function f' on S', and the range of the quantifiers would have been S'. Still, having made it we keep our initial stipulation *fixed* until further notice.

Based on the setup above such stipulations determine the referents for *all* our arithmetic terms; or at least they do so as long as we stick to our initial choice of a model. That is to say, relativist structuralism works with a notion of reference (modeled on the notion of interpretation in model theory) that is *relative* to such a choice – thus its name. On the basis of such reference it is also determined what is meant by "the natural numbers"; namely the particular model M of PA_2 that has been chosen initially. Of course this choice is largely arbitrary, since we could have picked any other model of PA_2 instead. But that does not matter. All that matters, from this point of view, is that we are *consistent* about our choice. As W. V. Quine notes in his article "Ontological Relativity":

> The subtle point is that *any* progression [i.e., natural number system] will serve as a version of number so long and only so long as we *stick to* one and the same progression (Quine 1969, 45, our emphasis)

In addition, we can now talk about *truth* in a determinate way as well. Namely, we can say that $p(1, s)$, thus p, is true if and only if $p(1, s)$ is true in the chosen model M (as defined along familiar Tarskian, model-theoretic lines).

But why does such relativity of reference not cause problems, in particular with respect to truth? The answer is, of course: because all models of PA_2 are isomorphic. Thus we will always agree on the truth value of a given statements in the language of arithmetic, no matter which model we have picked initially.[16] In other words, while truth has been *defined* in a relative way, a non-relative notion of "truth in arithmetic" is actually *implied*: truth in all models of PA_2. Analogously for the real numbers. Note, at this point, that in the case of group theory the situation is quite different. Here we do not arrive at the same non-relative notion of truth. More precisely, while we can still talk about those sentences in the language of group theory that are true in all groups, we cannot rely on one particular group to determine them.

Let us dwell a bit more on the core idea in relativist structuralism: its relative notion of reference. What a relativist structuralist does is, in a certain sense, to take arithmetic statements "at face value". That is to say, on the basis of the initial choice of a model '1' is treated as an object name (a singular term), i.e., as referring to a particular object; similarly, 's' is treated as a function name, i.e., as referring to a particular function; and variables are taken to range over a particular set of objects. However, in another sense arithmetic statements are not taken "at face value": not only does such reference always depends on an initial stipulation, we can also always switch things around by making a different stipulation. This last, variable aspect of relativist structuralism can perhaps be compared profitably to our ordinary use of indexical expressions, e.g., 'my car' or 'my house'. As with them we are here dealing with a case of *systematic referential ambiguity* – in a sense we are always talking about "my number 1", "my successor function", etc.[17] Note also, however, that in the present case it is not just ambiguity for one or a few related expressions, but for the whole language together.

Two other basic observations about relativist structuralism should be added. First, if we want to understand arithmetic along these lines, we *have to assume* the existence of an infinite set. Otherwise there simply is no model to pick, and the proposed semantics just runs empty – arithmetic terms have no reference, arithmetic sentences no truth value. Such an existence assumption is, thus, a necessary presupposition for relativist structuralism, or at least for its applicability. Second, a relativist structuralist *usually assumes* that some infinite set is given to us independently from arithmetic. More particularly, it is assumed that we can talk about such an infinite set, indeed a variety of such sets, independently from our use of arithmetic language. Otherwise the approach would be unmotivated – its main point is precisely to provide arithmetic with a semantics. Of course

both of these assumptions are natural and unproblematic if we work with *set theory* as part of our background theory. Thus in *ZFC* the axiom of infinity guarantees the existence of an infinite set; similarly the power set axiom, the axiom of replacement, etc. guarantee the existence of larger infinite sets. And we can make statements about these sets by means of our set-theoretic language, a language whose terms are taken to already have a reference.

For many working mathematicians, especially those presupposing *ZFC* as part of their background theory, a relativist structuralist approach will seem quite natural. Such mathematicians will construct not only various set-theoretic natural number systems, but also corresponding set-theoretic real number systems, complex number systems, etc. They will then single out one of these systems as "the natural numbers", another as "the real numbers", etc. As an explicit example from the mathematical literature, consider what Andrew Gleason, after describing the corresponding constructions in detail, writes in his *Fundamentals of Abstract Analysis*:

> [I]t does not make the slightest difference which simple chain [i.e., natural number system], complete ordered field, or complex number system we consider. If however, a reference to *the real number 1*, say, is to make sense, we must make a definite choice. A convenient choice is one which makes a real number just a special complex number. (Gleason 1991, 132).

Note here that Gleason is not a formalist, but a relativist structuralist. Note also that along these lines set theory functions as the *foundation* for all of mathematics in the following sense: All mathematical theories – except, of course, set theory itself – are to be treated in the relativist structuralist way described above. That is to say, models for all of them are constructed within set theory; and talk about reference, truth, "the natural numbers", "the real numbers", etc. is then taken to be relative to such constructions.

A relativist structuralist approach, in particular in combination with set theory, has several merits. One is that it allows for a comprehensive, unified treatment of many otherwise separate branches of mathematics: arithmetic, analysis, group theory, topology, etc. Another merit is that, as mentioned above, in set theory all the basic assumptions are made explicit and definite in terms of the axioms, including all the existence assumptions concerning relational systems. A number of contemporary philosophers of mathematics want to go a step further, though. They want to claim that the *real* merit of such an approach is one of "ontological economy". Consider the following remark by Quine:

> [To say] what numbers themselves are [along relativist structuralist lines] is in no evident way different from just *dropping numbers* and assigning to arithmetic one or another new model, say in set theory. (Quine 1969, 43–44, our emphasis)

The suggestion is this: A relativist structuralist approach with *ZFC* (or some equivalent theory) in the background allows us to restrict our "ontological commitments" in mathematics to one kind of entities only: sets. We don't need numbers in addition, as some other kind of mathematical entities, in order to understand what arithmetic is about; so "just drop them".[18]

This suggestion leads, however, directly to some new questions. Notice, in particular, that such an approach is *not* structuralist at one crucial point: the basic level of sets (which is why views like Quine's above are usually not called "structuralist", although they do deserve that name partly). How are we then to think about sets; do we have to accept them as a special kind of abstract objects, to be thought of in a "platonist" or "realist" sense; and if so, what exactly does that mean? Also, what is so special about sets that they deserve to be treated differently, i.e., granted some special, non-structuralist kind of reality? Put the other way around, if we can treat sets that way, why not the natural numbers, the real numbers, and other mathematical objects as well?

At this point in the discussion some philosophers of mathematics want to turn in a different direction: strict nominalists, e.g., Goodman, the early Quine, and more recently Hartry Field.[19] Such nominalists want to be even more economical in their ontology: they want to reject the appeal to *all* abstract objects, including sets. Interestingly, along such lines a relativist structuralist approach may still be attractive, if pushed a step further. The idea is this: Why not use only nominalistically acceptable objects, including mereological sums etc.[20], to form the basic relational systems we need? In other words, why not defend relativist structuralism *on a nominalist basis*?

Of course, such an approach raises several questions in turn. To begin with, if we want to be able to deal with arithmetic along these lines, we know that a model for it requires an infinite set, or sum, of objects as its basis; but where are we supposed to find these objects? Suppose the answer is simply: let's use physical objects. Then we are faced with the problem that the existence of infinitely many physical objects is not a trivial, unproblematic assumption, as the physical universe may be finite. Also, should we really have to rely on such empirical assumptions about the universe to ground mathematics? A modified nominalist answer might then be: let's use space-time points; or, along somewhat different lines, quasi-abstract objects such as strokes '|', '||', '|||', etc. However, arguably such entities bring with them their own peculiar problems; in fact, it is not clear that they are really better understood than the natural numbers themselves. In addition, note that we do not just need an infinite sum of objects to form a model of PA_2, we also need a successor function defined on it. How is this function to be understood now, if not along set-theoretic or similar "abstract" lines? In other words, is there a nominalistically acceptable way of dealing with the functions we need in arithmetic? Finally, all these questions become even harder to answer once we go beyond arithmetic, i.e., when we turn to parts of mathematics (including set theory itself) in which uncountable collections of objects, more complicated functions, etc. are at the center of attention.

We do not want to answer any of these questions here. Instead, reflecting on the corresponding versions of relativist structuralism – based on set theory or on some other restricted "ontological commitments" – we want to make the following more general observation: In many cases structuralist approaches in the philosophy of mathematics are pursued because they are taken to involve a kind of *eliminativism*. This is in particular true for relativist structuralism, and we can now clearly see why. Actually, there are two separable issues involved, or two aspects to the eliminativism: First, according to relativist structuralism we can restrict ourselves to *one kind of basic entities*, e.g., sets. Second and more subtly, we can account for arithmetic, say, without appealing to a *special, unique system*, "the natural numbers", distinct from all the other models of PA_2. In fact, even if there were such a special system it would not matter. All we are interested in, from this point of view, are models of arithmetic – *any* such models – just insofar as they are models. Consequently the assumption of a special system simply isn't needed (neither mathematically nor semantically or metaphysically); so apply *Occam's Razor* to it.

The second eliminativist aspect just identified is related to an additional "structuralist" argument that has some prominence in the literature. Remember, again, that there are various models of PA_2 in set theory; and from our current point of view all of them are equivalent. This equivalence has two sides: (i) any of these models is *capable* of playing the role of "the natural numbers"; (ii) none of them is *privileged* in this capacity. So far we have focused on the first side. But note how Paul Benacerraf uses the second in remarks such as the following:

> If numbers are sets, then they must be *particular sets*, for each set is some particular set. But if the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that this is an unknowable truth is hardly tenable. But there seems to be little to choose among the accounts. Relative to our purposes in giving an account of these matters, one will do as well as another, stylistic preferences aside. (Benacerraf 1965, 284–285, emphasis in the original)[21]

Similarly, Charles Parsons writes:

[I]f one identification of the natural number sequence with a sequence of sets or 'logical objects' is available, there are others such that there are no principal grounds on which to choose one. (Parsons 1990, 304)

In these passages both Benacerraf and Parsons address, or challenge, the question what the natural numbers "really are" or what they should be "identified with" in *an absolute sense.* Translated into our present framework the argument is then: None of our relative choices of set-theoretic models, say, is preferable to any other, at least beyond merely "stylistic" or "non-principled" considerations. But then we shouldn't consider any one of these models to be, in an absolute sense, "the natural numbers". To put the conclusion more briefly, if slightly misleadingly: in an absolute sense *there are no natural numbers.*[22]

Coming back to our general discussion, we have seen that relativist structuralism is not without its difficulties. In particular, it requires either a separate treatment of set theory or, alternatively, the assumption of infinitely many, continuum many, etc. other basic objects, as well as a corresponding account of functions. Let us add one final comment about relativist structuralism. Remember that according to this position we initially pick one model of, say, arithmetic and we then stick to that model. Note that, later on, we *never* have to appeal to any properties peculiar to the objects in that model, at least not as far as arithmetic is concerned. In other words, we can "abstract away" from all such properties, in the sense of ignoring them completely – that is, in fact, the main sense in which this view is "structuralist" (compare guiding idea (2) from Section 1). But if that is the case, couldn't we instead understand arithmetic sentences to be about *all* models of PA_2 at the same time, not about *any* one of them relatively speaking? Such a shift – the move from "any" to "all" – leads to a variant of structuralism that deserves separate treatment.

5. UNIVERSALIST STRUCTURALISM (INCLUDING MODAL VARIANTS)

According to relativist structuralism a term like '1' is understood to refer to a particular object, namely to the base element of some chosen model of PA_2. In that sense '1' is treated as a singular term, even if a somewhat unusual, relative one. In contrast, according to *universalist structuralism* we want to treat '1' as referring, somehow, to *all* base elements of models of PA_2 at the same time; similarly for 's' etc. Is there a way to make this basic idea more precise?

In order to be both precise and explicit here, let us start with a slight modification of the way in which the Dedekind–Peano Axioms are formulated. Instead of using two primitive, non-logical symbols, '1' and 's', we use three, i.e., we add a one-place predicate symbol 'N' (where "$N(x)$" is to be understood as "x is a number", or better "x is a natural number object"); we also *restrict* all the quantifiers involved to N. This means that our axioms for arithmetic look as follows:

(A1′) $N(1)$,

(A2′) $\forall x[N(x) \rightarrow N(s(x))]$,

(A3′) $\forall x[N(x) \rightarrow (1 \neq s(x))]$,

(A4′) $\forall x \forall y[((N(x) \wedge N(y)) \wedge (x \neq y)) \rightarrow (s(x) \neq s(y))]$,

(A5′) $\forall X[(X(1) \wedge \forall x((N(x) \wedge X(x)) \rightarrow X(s(x))))$
$\qquad \rightarrow \forall x(N(x) \rightarrow X(x))]$.

Let $PA_2(1, s, N)$ be the conjunction of these five axioms. We can now, once more, consider various systems that satisfy this condition, i.e., models of $PA_2(1, s, N)$, where such models consist of a set S (the general domain), a distinguished element e in S (corresponding to '1'), a one-place function f on S (for 's'), and a subset S' of S (for 'N').

Let p be an arbitrary sentence of arithmetic. Our basic question is: How should we understand what p is about? More particularly, do any of the terms in it refer; if so, to what; and what do the quantifiers range over? In the case of relativist structuralism we introduced a corresponding sentence $p(1, s)$, with only '1' and 's' as primitive symbols, when answering that question. In the present case we proceed in a slightly more complicated way, consisting of three steps: First, parallel to the move from $PA_2(1, s)$ to $PA_2(1, s, N)$ we translate p into a sentence in which only '1' and 's' are used as *primitive symbols* and in which all the occurrences of *quantifiers* are *restricted* to N (see, e.g., (A4′) above). Let $p(1, s, N)$ be the resulting sentence. Second, rather than working directly with this sentence we introduce an *if-then* statement containing it, namely:

$$PA_2(1, s, N) \rightarrow p(1, s, N).$$

Third, we *quantify out* the terms '1', 's', 'N' so that we end up with the following:

$$\forall x \forall f \forall X[PA_2(x, f, X) \rightarrow p(x, f, X)].^{23}$$

In this last sentence we use, as usual in 2nd-order logic, three kinds of variables: 'x' for objects, 'f' for one-place functions, and 'X' for one-place predicates or sets. Altogether, let q be the universal if-then statement we have just constructed out of p.[24]

This construction, or translation, puts us in a position to make clear what the core of universalist structuralism is. It is again a *semantic thesis*, namely the following: Whenever we use an arithmetic sentence p to assert something, what we really assert is a universal if-then statement, as made explicit in q. Several basic aspects of this thesis should be pointed out at once. First, note the specific *if-then* character of q, i.e., the way we have built a material conditional right into it. This aspect distinguishes universalist structuralism immediately from relativist structuralism. Second, note that we again "abstract away" – now by *generalizing* – from what is peculiar about any particular model of PA_2. That is the main sense in which the position is "structuralist" (compare again intuitive thesis (2)). In fact, third, any reference to specific models of PA_2, or to particular objects and functions in them, has disappeared completely (even in a relative or model-theoretic sense). Instead, what we assert with an arithmetic statement p is now something about *all* objects, *all* one-place functions, and *all* one-place predicates or sets; since the main logical operators in q are unrestricted universal quantifiers.

This third aspect of universalist structuralism, which makes it really "universalist", may seem odd at first. Note that, along these lines, even a sentence like '2 + 3 = 5' is used to make not a particular, but a universal statement. A universalist structuralist is willing to bite that bullet; in fact, it is seen as exactly appropriate for mathematics. Here is how Bertrand Russell, an early defender of such a view (although not under this name), endorses it in his article "Recent Work in the Philosophy of Mathematics":

> Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. (Russell 1901, 76–77, emphasis in the original)

Similarly Russell's *Principles of Mathematics* starts with the following declaration:

> Pure mathematics is the class of all propositions of the form "p implies q", where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. (Russell 1903, 3)

(As becomes clear later on in *Principles*, the variables in "propositions of the form 'p implies q' " have to be understood as universally quantified and the "implies" as the material conditional. Thus all the ingredients of universalist structuralism are present.)

Relativist and universalist structuralism are obviously related to each other. To clarify their relation further we can briefly consider a proposal that is half way between the two, although this proposal is probably not stable in the end. Going back to PA_2 and $p(1, s)$ one may want to suggest this: Why not say that in asserting $p(1, s)$ we talk about the various models of PA_2 *all at once*, not just one at a time as in relativist structuralism? That is to say, why not stipulate that the constant '1' in $p(1, s)$ refers to *all* the things that function as base elements in the various models, not just to a particular one; similarly for 's' etc. (where the reference of '1', 's', and everything defined in terms of them is again understood as "coupled together")? In other words, why not adopt the universalist aspect of universalist structuralism while ignoring its if-then aspect? The result is, however, the following: '1' is now supposed to *refer to all objects whatsoever* at the same time, since any objects can play the role of the base element in a model of PA_2. That seems an odd kind of reference, if it can be worked out in a coherent way at all.

A good way to think about universalist structuralism is that it avoids this oddity while preserving the universalist idea motivating such a proposal. Again, that result is achieved by introducing a universal if-then sentence q corresponding to each arithmetic sentence p. This brings us to a fourth basic aspect of universalist structuralism that needs to be made explicit. It concerns the nature of the relation between p and q. Note that, along the lines above, the newly introduced sentence q is meant to make *explicit* what was, in some sense, already *implicit* in p. We can reformulate this point slightly to bring out its real force: According to universalist structuralism every sentence p as used in arithmetic has a certain "surface form", namely its usual *syntactic form*; but it also has a "deep form", what one may call its *semantic* or *logical form*; the latter is what needs to be laid bare to see what we *really mean* when we use p in arithmetic; and it is laid bare in terms of the syntactic form of q.

In universalist structuralism we are, thus, not taking arithmetic language "at face value", not even in the relativist structuralist sense. Rather, we *analyze* every arithmetic statement p in a non-trivial way, as reflected in the syntactic form of q. Actually, it is possible to modify and weaken this thesis somewhat, by appealing to the notion of *explication* (in a Carnapian sense) instead of *analysis*. The modified claim will then be this: While q does not make explicit what was already implicit in p, it is related to p in the sense of explicating it, i.e., of clarifying its content, sharpening it, and in the end replacing it. But even in this modified form, it seems that p and q have to be related in some intrinsic way for the view to have *philosophical* significance.[25]

Once more, the basic move in universalist structuralism is to replace p by q. What are the consequences of that move with respect to *truth* in arithmetic? Well, the truth of p is simply understood in terms of the truth of q; and for q to be true something has to hold for all objects, for all one-place functions, and for all one-place predicates or sets. Put that bluntly, this looks like a very radical, revisionist suggestion. In the end it is, however, not so different from a relativist structuralist view. It still holds – because of the universal if-then form of q – that p is true if and only if $p(1, s, N)$ is true in all models of $PA_2(1, s, N)$ (in the model-theoretic sense); and the latter is basically equivalent to $p(1, s)$ being true in any, thus in all models of $PA_2(1, s)$. (At least it is equivalent if we presuppose that there are such models, a necessary presupposition for relativist structuralism anyway.)

Like relativist structuralism, universalist structuralism can be seen as a form of *eliminativism*, again in two ways: first, as an eliminativism directed against the unnecessary postulation of abstract objects *in general*, with the goal of eliminating their use as much as possible;[26] second, as an eliminativism in which the assumption of a *special, unique* system of objects, to be identified as "the natural numbers", is avoided or "erased". At the same time, the exact form this erasure now takes is interestingly different from relativist structuralism. Instead of treating '1' as an ambiguously referring expression we now treat it as a *variable*. Or more precisely, the constant '1' is *quantified out* in the move from p to the more complicated formula q; similarly for 's'.

Universalist structuralism, like relativist structuralism, is not without its difficulties. We just noted that every arithmetic sentence p turns out, in its analyzed or explicated form, to amount to a universally quantified sentence q. That aspect may at least seem surprising. But the main problem arises when we ask: What is the *range* of the three universal quantifiers, or of the corresponding variables, in q supposed to be, respectively? Several answers may be suggested in response. The most direct and simple answer is: let x range over *all objects*; let f range over all first-level, one-place functions; and let X range over all first-level, one-place predicates or sets (along the lines of a Fregean "universalist" conception of logic). Actually, to avoid Russell's antinomy we need to be more careful with what is meant by "all objects" here, i.e., we should add type restrictions along Russellian lines (or some corresponding safeguard). Let us say, then, that x ranges over all objects that are themselves not sets, i.e., all objects of lowest type, etc.

In this case we have to admit, right away, that some abstract entities have not been eliminated after all: sets and functions of objects of lowest type. But even if that is accepted as unavoidable, we are confronted with another difficulty, also already noted by Russell: what if there exist only finitely many objects of lowest type? If so, then there simply are no models of the Dedekind-Peano Axioms (either in the form $PA_2(1, s)$ or $PA_2(1, s, N)$), or at least there are no models built up out of the right kind of objects. Note what that implies in our present context: *all* our arithmetic statements turn out to be *true*, since all of them have turned into universal if-then statements whose antecedents $PA_2(x, f, X)$ are never satisfied. In other words, all arithmetic statements, even something like "1 = 2", turn out to be *vacuously true* – clearly not a result that is acceptable. This is a serious problem for universalist structuralism, accordingly called the *non-vacuity problem*.

What is the right response to the non-vacuity problem? Several suggestions may again be considered. The most straightforward is to assume an *axiom of infinity* for the lowest type of objects, à la Russell. But with what justification, e.g., as an empirical claim? Alternatively we can take a *modal turn*. This can be done in at least two different ways. First, instead of assuming that we quantify over all actual objects, why not quantify over all *possible objects*, i.e., why not conceive of our basic domain of quantification in such a broader way? However, that leaves us with many tricky questions about such *possibilia*, including whether there are, in some sense, enough of them available. Also, doesn't it go directly against the eliminativist intent which usually motivates a universalist structuralist approach?

Instead we can "go modal" in a second way: we can add a *necessity operator*, a box '$\Box$', in front of our translations q. More explicitly, instead of using q as the analysis of p we now use $\Box q$. All arithmetic sentences then turn out to have the following form:

$$\Box[\forall x \forall f \forall X (PA_2(x, f, X) \rightarrow p(x, f, X))].$$

This move leads directly to the position worked out, in great detail, in Geoffrey Hellman's writings.[27] Given our general framework it is clear what we have arrived at: a version of *modalized universalist structuralism*; or more briefly: *modal structuralism*.

Suppose we follow Hellman's modal route. Are we then "home free" as far as our earlier difficulties go? Not entirely. First, we are still left with a version of the non-vacuity problem, although perhaps a weaker one. It has this form: Is it *possible* to *satisfy the antecedent* of q, i.e., is $\Diamond \exists x \exists f \exists X PA_2(x, f, X)$ true; or equivalently (given the definition of $PA_2(x, f, X)$), is the existence of an infinite set of objects possible? (It is not hard to see, with a little bit of modal logic, that we need to assume so, since otherwise all our sentences $\Box q$ turn out to be vacuously true again.)

But then, is a positive answer to that question really easier to justify than an axiom of infinity itself, or than the postulation of an infinity of numbers or sets seen as abstract objects? That depends on how we interpret '$\Box$' and '$\Diamond$', i.e., what kind of modality we adopt. More particularly, it depends on what we take the necessary and sufficient conditions for the relevant kind of possibility to be. Logical consistency is almost certainly a necessary condition. But is it also sufficient? If not, what else is involved? As earlier, such questions become even more pressing if we go beyond arithmetic, i.e., turn to parts of mathematics in which higher infinities are involved, starting with analysis.[28]

Second, the following challenge can also be brought up: Is it really so plausible to take a sentence of the form $\Box q$ to reveal what is always already implicit in p? In other words, do our ordinary arithmetic sentences really have such a complicated semantic or logical form hidden underneath their usual syntactic form, including a modal component? That seems even more questionable, or at least in need of further justification, than if we just appeal to q. A third, related problem is this: Note that along modal-structuralist lines mathematical truth turns into a kind of modal truth: p is true if and only if $\Box q$ is true, i.e., if q is necessary. But is that really an attractive view about mathematical truth? In response it might be said that mathematical truths do, indeed, carry a kind of necessity with them, a necessity nicely captured by modal structuralism, thus actually motivating it further. Then again, at least contemporary mathematical practice, especially as guided by the structuralist methodology, doesn't seem to involve modality in any direct way, does it?[29]

So far we have considered adding an axiom of infinity, on the one hand, and two ways of "going modal", on the other, as responses to the non-vacuity problem. But some other responses are possible, too, still within the general framework of universalist structuralism. For instance, we can try, once more, to appeal to space-time points in this context, i.e., assume that they are included in our basic domain of quantification. Similarly for quasi-abstract objects such as strokes etc. In other words, we can try to defend universalist structuralism *on nominalist or quasi-nominalist grounds*.[30] Then again, along such lines we quickly encounter familiar questions (see above). Also, in each case we now encounter a corresponding version of the non-vacuity problem.

Finally, what about working with *set theory* in the background? In other words, can't we combine a set-theoretic ontology with a universalist structuralist semantics? In that case it will be the axioms of set theory, say *ZFC*, that guarantee non-vacuity. In fact, within a set-theoretic framework we can simplify our approach significantly: we can treat functions and relations as sets of ordered pairs, as usual; thus we can (with a few obvious modifications in the formulation of q) work with only one kind of variable. As a consequence all our quantifiers can be taken to range over one domain: the universe of sets. Of course we have then not eliminated *all* abstract objects. But we have, once again, restricted ourselves to one kind only. Similarly, this approach allows for a unified treatment of almost all parts of mathematics, including arithmetic, analysis, and group theory. It doesn't answer the question of how to think about sets themselves, though. And if we treat sets in an essentially non-structuralist way, the corresponding position will be "structuralist" only up to a point (likewise for a Russelian type-theoretic approach).

6. STRUCTURES: FROM PARTICULARS TO UNIVERSALS

So far we have discussed three main variants of structuralism: formalist, relativist, and universalist structuralism (including modal sub-variants). Looking back now, how do they compare to each other; and what is their connection with mathematical practice?

As should have become clear, all three of these positions are compatible with, even motivated by, our structuralist methodology. Both relativist and universalist structuralism then add a substantive *semantic* thesis to it, in each case with interesting *metaphysical* consequences: eliminativist etc. Also, both of them can be combined, at least in principle, with various basic *ontologies*: from set-theoretic platonism to physicalist nominalism. This is in sharp contrast to formalist structuralism, whose response to the corresponding semantic and metaphysical questions is much more negative or deflationary. At the same time, there is an interesting contrast between the semantics at the core of relativist and universalist structuralism, related as they are. In relativist structuralism the basic semantic idea is to pick a particular model of the relevant theory and to explain the reference of mathematical terms with respect to it; in universalist structuralism the appeal to particular objects and the use of referring terms is simply quantified away.

Let us go back to our initial, vague characterizations of "structuralism" (in Section 1) in order to emphasize certain similarities and differences further. Note that according to all three variants of structuralism discussed so far our two initial intuitive theses hold: (1) that mathematics is primarily concerned with the "investigation of structures"; and (2) that doing so involves an "abstraction from the nature of individual objects". However, the *abstraction* in thesis (2) is conceived of in importantly different ways in the three cases. In formalist structuralism it is done by denying that there are mathematical objects in any substantive sense; in relativist structuralism by picking an arbitrary model of a theory and ignoring any additional properties the objects in it may have; and in universalist structuralism by talking about all objects at the same time, not about any particular ones.

At this point in our discussion the appeal to *structures* in (1) also needs to be addressed more directly. So far we have relied heavily on *relational systems* in this connection, usually thought of in a set-theoretic sense, sometimes in terms of nominalist variants. Such systems – e.g., a set S with a distinguished element e in it, a one-place function f on it, such that the triple satisfies PA_2 – are customary objects of study in contemporary mathematics. In addition, it is exactly *such* systems that are often called "structures" in the literature, both in philosophy and in mathematics. As an illustration from philosophy consider the following remark by Charles Parsons:

> What is meant by a structure is usually a domain of objects together with certain functions and relations on the domain, satisfying certain general conditions. Paradigm examples of structures are the elementary structures considered in abstract algebra. (Parsons 1990, 305).[31]

With respect to the mathematical literature, compare this passage from Saunders Mac Lane's *Mathematics: Form and Function* in which the focus is on examples from algebra:

> These [particular semi-groups, groups, rings, etc.] and many other cases illustrate the general notion of an algebraic structure: A set X with nullary, unary, binary, ternary ... operations satisfying as axioms a variety of identities between composite operations (Mac Lane 1986, 26)

As is implicit in Mac Lane's remark, mathematicians often sub-divide the whole class of such structures further: into "algebraic structures", "order structures", and "topological structures"; and there are many "mixed structures" as well.[32]

For the discussion in the rest of this paper it will be useful to have another, more distinctive name for "structures" of this kind, i.e., for relational systems in the sense discussed so far. Let us call them *particular structures*, since they consist of particular relational systems. Now, there is also a second, quite different usage of "structure" in the literature, again both in philosophy and in mathematics. This second usage is seldom explicitly distinguished from the first, although it is probably just as common. According to it a "structure" is *not* a particular relational system. Rather it is what various such systems *share* (or what they don't share); put differently, it is what is (or isn't) *instantiated* by them. That is to say, a "structure" is now a *universal*, not a particular.[33]

Such talk of "sharing", "instantiation", and "universals" may remind us of the old (Platonic/Aristotelian) distinction between "matter" and "form". We may thus be led to the thesis, sometimes expressed by mathematicians themselves, that mathematics is the study of form, not of matter. As Henri Poincaré puts it in *Science and Hypothesis*:

> Mathematicians do not study objects, but the relations between objects; to them it is a matter of indifference if these objects are replaced by others, provided that the relations do not change. *Matter* does not engage their attention; they are interested in *form* alone. (Poincaré 1905, 20, our emphasis)

Of course, when mathematicians like Poincaré use expressions such as "matter" and "form" in such passages, similarly when they use "structure" in our second sense, they typically do so in an informal, loose way. In particular, they do not attempt to clarify what exactly the corresponding form or structure now *is*, including what *kind of entity* it might be. To do so is not necessary for their mathematical purposes. For our philosophical purposes, on the other hand, answering such questions is important. Let us then give the corresponding "structures" a new name as well; let's call them *universal structures*. What we want to do in the next two sections is to explore further how one could and should think about them.

7. PATTERN STRUCTURALISM

If we compare two different models of PA_2, say, what is it that they share? Well, they are both models of these axioms, and this implies that they are isomorphic (since PA_2 is categorical). In each of them there is, thus, a distinguished base element, then its immediate successor, then its successor, etc., and by iterating this process we reach all its elements. We can, as it were, lay the two models alongside each other and see that they "look" the same – they exemplify the same "pattern". To exploit this image a bit further, with respect to such a pattern we can distinguish various "points", "positions", or "roles": one corresponding to all the base elements, one corresponding to their immediate successors, etc. In a particular model of PA_2, on the other hand, its elements "occupy" the respective "points" or "positions", or they play the corresponding "roles".

So far we have talked loosely and figuratively. If we take such talk about patterns, positions, etc. more seriously – if we articulate them as substantive semantic and metaphysical theses – we are led to *pattern structuralism*. According to this position what we really study in arithmetic, in the end, are not the various particular models of PA_2, but something in addition to them: a corresponding *pattern*. In the words of Stewart Shapiro, one of the main proponents of such a view:

The subject matter of arithmetic is a single *abstract structure*, the *pattern* common to any infinite collection of objects that has a successor relation with a unique initial object and satisfies the (second-order) induction principle. (Shapiro 1997, 72, our emphasis)

Note that the natural number pattern, or structure, appealed to here is meant to be different from all the relational systems corresponding to it. It is, in fact, a *new kind of abstract entity*, as we will see more clearly soon.

Three basic aspects of patterns, as understood by a pattern structuralist in our sense, need to be made explicit immediately. First, such patterns can, and usually do, have many different *instantiations* or *exemplifications*. We have already seen, implicitly, that the natural number pattern is instantiated in set theory by the finite Zermelo ordinals, the finite von Neumann ordinals, etc. In addition, Shapiro gives the following examples:

The natural-number structure is exemplified by the string on a finite alphabet in lexical order, an infinite sequence of strokes, an infinite sequence of distinct moments in time, and so on. (*ibid.*, 73)[34]

It is in this sense that a pattern is a *universal*; or as Shapiro puts it:

(A) structure is a one-over-many. [...] Thus, structure is to structured as pattern is to patterned, as universal is to subsumed particular, as type is to token. (*ibid.*, 84)

Note that, as such a universal, a pattern is different not only from particular relational systems, but also from all other objects (as usually understood).

Second and in addition, patterns are supposed to have a special kind of *internal composition*: they consist of "positions", "points", or "nodes" related to each other in a certain way. In this respect they differ not only from ordinary objects, but also from other, more traditional universals (more on that below). Third, the identity and the nature of the positions in a pattern depend solely on their being part of that pattern, nothing else. In a sense there *is* nothing more to them; or as Michael Resnik, another defender of pattern structuralism, puts it (in a passage quoted approvingly by Shapiro):

The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are themselves atoms, structureless points, or positions in structures. As such they have no identity or distinguishing features outside a structure. (Resnik 1997, 201)

Remember here also Charles Parsons' characterization with which we started this paper:

By the "structuralist view" of mathematical objects, I mean the view that [...] the objects have no more to them than can be expressed in terms of the basic relations of the structure. (Parsons 1990, 303)

Note that a pattern structuralist thus explicitly subscribes to our initial thesis (3), that mathematical objects "have no more to them than can be expressed in terms of the basic relations of the structure".

Some philosophers with general sympathies to pattern structuralism worry exactly about the third aspect just mentioned, i.e., the identity and nature of positions in a pattern. Thus Paul Benacerraf writes in connection with the natural number pattern:

Therefore, numbers are *not objects at all*, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure* – and the distinction lies in the fact that the 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure. (Benacerraf 1965, 291, our emphasis)

About ordinary objects he remarks, in contrast:

That a system of *objects* exhibits the structure of the integers implies that the elements of that system have some properties not dependent on structure. It must be possible to *individuate* those objects independently of the role they play in that structure. (*ibid.*, our emphasis)

Benacerraf's worry can be put this way: Positions in patterns are distinguished by their systematic, ungrounded interdependency, i.e., by the fact that they have "no properties other than those relating them to other 'elements' of the same structure". But does that not exactly rule them out as *objects*? Or stronger, are they then *acceptable entities* at all?

Benacerraf's concern is similar to one voiced by Bertrand Russell earlier, already in his *Principles of Mathematics*. Objecting basically to a pattern structuralist view about the natural numbers, which he attributes to Dedekind, Russell remarks:

[I]s is impossible that the [numbers] should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression [i.e., a natural number system]. If they are to *be anything at all*, they must be *intrinsically something*; they must differ from other entities as points from instants, or colours from sounds. (Russell 1903, 242, our emphasis)

The core of Russell's objection is this: If positions in a pattern are not "intrinsically something", i.e., if they lack the characteristics that individuate ordinary objects, then they "are nothing at all", i.e., they can't possibly exist.

How exactly we should understand this Russellian objection, as well as its precise relation to Benacerraf's, depends on how we understand Russell's distinction between "intrinsic" and "extrinsic" properties. We want to leave this interpretive question aside. Let us just observe that recent pattern structuralists such as Resnik and Shapiro are not deterred by this objection, either in Benacerraf's or Russell's form. Their response is the following: If we work with a notion of "object" that is motivated by and in the end restricted to ordinary objects, in particular to physical objects and perhaps sets (in the usual sense), then of course positions in patterns are not objects. But why not simply broaden our notion of "object" in a certain way?

Shapiro explains this broadening by distinguishing two perspectives one can take with respect to a given pattern, or better with respect to the positions or places in it: (a) the "places-as-offices" perspective where we treat these places just as slots to be filled by ordinary objects; (b) the "places-as-objects perspective" where we consider them in themselves, i.e., refer to them with singular terms, talk about their relations to other places, etc. He adds:

> Arithmetic, then, is about the natural-number structure, and its domain of discourse consists of the places in the structure, treated from the places-as-objects perspective. The same goes for the other non-algebraic fields, such as real and complex analysis, Euclidean geometry, and perhaps set theory. [...] When the structuralist asserts that numbers are objects, this is what is meant. (Shapiro 1997, 83)

For a pattern structuralist positions in patterns are, thus, objects in a weak sense – a sense exactly appropriate for mathematics, as both Resnik and Shapiro would insist.

Before reflecting further on this conception of positions and patterns, let us observe four immediate consequences of pattern structuralism as described so far. First, this position is clearly *not* an *eliminativist* position in the sense of trying to do without abstract objects as much as possible. In fact, its very core consists of the postulation of a new kind of abstract entities, patterns, over and above ordinary physical objects, set-theoretic objects (as usually understood), etc. Moreover, for Shapiro patterns or universal structures are supposed to exist *prior to* and *independently of* their instantiations. In his own words:

> Structures exist whether they are exemplified in a nonstructural realm or not. On this option, statements in the places-are-objects mode are taken literally, at face value. (*ibid.*, 89)

This is why Shapiro calls his position "*ante rem* structuralism" (and "*ante rem* realism", as opposed to "*in re* realism), in analogy to a traditional view, attributed to Plato, concerning universals more generally (*ibid.*, 40–41).[35]

Second, Shapiro's view is also not eliminativist in our second sense above. According to him there exists a *special, unique* entity that deserves the name "the natural numbers", namely the natural number pattern (more on its identity below). Third, given this natural number pattern we can explain *reference* in arithmetic in a relatively straightforward way: '1' now refers to the initial position in the pattern, '*s*' to the successor function on the pattern, and our variables range over the pattern. Fourth, *truth* can be understood correspondingly, i.e., as truth-in-the-pattern (understood along Tarskian lines, based on such reference). Of course, an arithmetic sentence

is true in this sense if and only if all the "translations" corresponding to its various instantiations are true as well, i.e., if and only if the sentence is true in all models of PA_2 (in the usual model-theoretic sense).

Such explanations of reference and truth in arithmetic depend, however, on the assumption that there is a unique, special natural number pattern in terms of which they can be defined. This assumption is defended in detail by Shapiro – but it depends on a certain decision that is resisted by other pattern structuralist, e.g., Resnik. Shapiro's decision is to rely on a specific notion of *pattern identity*, i.e., a criterion that allows us to decide whether two patterns are identical or not. Resnik's resistance is based on the view that there is "no fact of the matter" with respect to such identities, both as far as patterns and as far as positions across patterns are concerned. What that means for Resnik comes out in passages such as the following:

> Number theory, for example, is intended to deal with a certain structure; it has the means to raise and answer questions concerning the identity of various numbers, but it cannot even formulate the questions as to whether the number one is the real number *e*. [...] Each theory [arithmetic, real analysis, set theory] was developed to speak only of elements of a certain structure and has no means to identify or distinguish these from elements of another structure. (Resnik 1997, 211)

Resnik's view is this: There is "a fact of the matter" concerning a scientific question if the relevant science has in principle a means to determine its answer. But that is exactly not the case here: mathematics has no means to answer questions about the identity or difference of the various patterns it deals with, nor about the identity or difference of positions across patterns (as opposed to within a given pattern).

What is at issue here can be illustrated as follows: Assume that we are given a natural number pattern and a real number pattern, as understood by a pattern structuralist. Then there exists a certain "subpattern" of the real number pattern that is isomorphic to the natural number pattern, namely that formed by "the real number one", "the real number two", etc. Now, is this subpattern *identical* with the given natural number pattern or not? Relatedly, is "the (natural) number one" in the original natural number pattern *identical* with "the (real) number one" in the new subpattern or not? Resnik's point is that neither arithmetic nor analysis can give us an answer. Similarly if we consider natural number subpatterns in set theory; in Resnik's words again: "Nothing in science or mathematics, as ordinarily understood, counts as evidence for or against numbers being sets" (*ibid.*, 246). He refers to this aspect in general as the "incompleteness of mathematical objects" (*ibid.*).

Initially Resnik's position here might look attractive. But note what the direct consequences for a pattern structuralist are: If there is no fact of the matter as to whether there exist one natural number pattern or several different ones, then there is also no fact of the matter what '1', 's', etc. refer to. In other words, the explanation of reference above, thus also that of truth, can't get a foothold. Resnik acknowledges this point:

> But if there is no fact of the matter as to whether the positions in a pattern are the same or distinct from those in one of its occurrences, then there is none as to whether general or singular terms refer to the positions in the one rather than the positions in the other. (*ibid.*, 220)

On the other hand, he writes:

> My claim is that there is enough slippage between our theories of patterns and the patterns themselves to affect reference. But it does not affect truth. For the truths of a theory of a pattern are invariant under all reinterpretations in patterns congruent in it. (*ibid.*, 222)

At this point it is not clear, however, how exactly we are to think about truth now. Perhaps Resnik has to fall back on ideas similar to those in relativist structuralism at this point, i.e., on a relative notion of reference and a corresponding relative notion of truth.

Actually, sometimes Resnik moves even further away from the pattern structuralism exemplified in Shapiro's work, e.g., when he allows himself to talk about "the group structure", "the ring structure", or "the first-order natural number structure" (*ibid.*, 252). In these cases we can find models for the respective theories that are not only not identical, but also not isomorphic. But in what sense is there then *one* corresponding structure, so that the definite article "the" is justified? A pattern structuralist view seems, thus, misplaced with respect to such cases (although we can still talk about "structure" in a looser, more informal sense, as many mathematicians do).[36] For this reason Shapiro, in contrast to Resnik, distinguishes between "algebraic theories", such as group theory and ring theory, and "non-algebraic theories", such as arithmetic, analysis, and (to some degree) set theory. And he wants to restrict a pattern structuralist approach to the latter (Shapiro 1997, 40–41).

This still leaves Shapiro with the task of deciding about a notion of pattern identity for the "non-algebraic" case.[37] He considers two candidates: (a) *isomorphism* of the corresponding instantiations (b) *structure-equivalence* (a notion originally introduced by Resnik). The former criterion should be clear after our earlier preparations. The latter is defined as follows:

> First, let R be a system and P a subsystem. Define P to be a *full subsystem* of R if they have the same objects (i.e., if every object of R is an object of P) and if every relation of R can be defined in terms of the relations of P. The idea is that the only difference between P and R is that some definable relations are omitted in P. So the natural numbers with addition and multiplication are a full subsystem of the natural numbers under addition, multiplication, and less-than. Let M and N be systems. Define M and N to be *structure-equivalent*, or simply *equivalent*, if there is a system R such that M and N are each isomorphic to full subsystems of R. (Shapiro 1997, 91, emphasis in the original)

If we use this second criterion, we treat two patterns as identical if all their instantiations are structure-equivalent, in the sense just defined. If we use the first, we treat them as identical if all their instantiations are isomorphic.

Note that both of these criteria rely on equivalence relations for instantiations of patterns. In addition, the first is more fine-grained than the second, since it is more sensitive to variations in the language used, e.g., to whether we use "less-than" in arithmetic as primitive or as defined. Now, either criterion *could* be adopted by a pattern structuralist. Shapiro *prefers* to adopt the first one, based on isomorphism. He does so not because it is "the right one", but because it has certain pragmatic advantages. As he puts it, it is "technically inconvenient" to use the second (*ibid.*, 93). In any case, picking either of them is a decision, not a discovery; that much has to be granted to Resnik. Still, it makes sense to make such a decision, justified by pragmatic arguments. Based on it we can, then, talk about "the natural number pattern", "the real number pattern", etc.

No matter which of the two criteria we use, it follows that the identity of patterns is thoroughly *language-relative*. That is to say, which patterns are considered identical depends, in one way or another, on what our chosen language allows us to say (what it allows us to consider as an isomorphism, to define, etc.). Shapiro explicitly acknowledges this point when he writes: "In mathematics, at least, the notions of 'object' and 'identity' are unequivocally but thoroughly relative" (Shapiro 1997, 80). Neither Shapiro nor Resnik considers such language-relativity to be a real problem. For both it is simply an integral part of mathematics. Resnik goes as far as suggesting: "[T]here is no reason why a theory of structures could not recognize all L-structures for all choices of [languages] L" (Resnik 1997, 254). Shapiro concurs, although he does want to restrict himself to structures corresponding to categorical theories.

Even if we agree that such language-relativity is no problem, there are other difficulties that remain concerning pattern structuralism. One remaining question is this: How exactly are we to think about the relation between patterns and the positions in them? Presumably it is not that between sets and their elements, i.e., elementhood. But what is it then, perhaps that between part and whole; or is it different entirely?[38] Second, there is the question of how to think about functions and relations as defined on patterns. As we want, presumably, again not to rely on a set-theoretic understanding of them, do we have to take such functions and relations as primitive? If so, what does that entail? A third, perhaps more pressing question is the following: How do we decide which patterns *exist*? Initially we might want to answer that question by simply pointing to exemplifications of patterns in non-structural realms. But note, again, that most of the patterns we study in mathematics are infinite, e.g., the natural number pattern and the real number pattern. It is a question then, where a pattern structuralist can find corresponding non-structural exemplifications.

This brings us back to some old proposals and corresponding problems. One such proposal is to rely on exemplifications involving physical objects. This makes mathematics once more depend on assumptions about the physical world, not an appealing result for most pattern structuralists. Or we can appeal to space-time points, to quasi-abstract objects like strokes, etc., with all the corresponding questions those appeals raise in turn. Alternatively we can bring in set theory again. More precisely, we can take set-theoretic satisfiability of the corresponding theories as the criterion for pattern existence. As Shapiro himself notes, this would actually be in line with standard mathematical practice, since "in mathematics as practiced, set theory (or something equivalent) is taken to be the ultimate court of appeal for existence questions" (Shapiro 1997, 136). But then we will not be able to deal with set theory itself along pattern structuralist lines, on pain of circularity, something most pattern structuralists would want to be able to do.

If all of these alternatives seem unattractive, there is one further possibility: we can try to develop a systematic *pattern theory*. Such a theory should tell us axiomatically which structures exist, analogously to how *ZFC* set theory tells us which sets exist. It should also shed some light on our earlier questions: about the pattern-position relation, about functions, relations, etc. In fact, this is Stewart Shapiro's route, i.e., he proceeds to sketch exactly such a pattern theory in his book *Philosophy of Mathematics. Structure and Ontology* (especially in chapter 3). It turns out to be a theory closely modeled on set theory. Thus it includes an axiom of infinity, axioms corresponding to the subset, powerset, replacement axioms, etc. Beyond that, it includes axioms for "subtraction", "addition", etc. which do not have direct set-theoretic correlates.

Such a proposal leads, however, to a new question: Why duplicate set theory with such a structure theory, in particular if they turn out to be so similar? In other words, why not simply work with set theory itself, since we already know how to do so? Shapiro comments in this connection:

> Anything that can be said in either framework can be rendered in the other. Talk of structures as primitive is easily 'translated' as talk of isomorphism or equivalence types over a universe of (primitive) sets. In the final analysis, it does not really matter where we stand. (*ibid.*, 96)

He also claims:

> The fact that any of a number of background theories will do is a reason to adopt the program of *ante rem* structuralism. *Ante rem* structuralism is more perspicuous in that the background is, in a sense, minimal. (*ibid.*)

This leaves us with the question: In which sense exactly is the background of pattern theory supposed to be "minimal"; and why should such minimality be crucial? Finally, it is hard to imagine that something like Shapiro's structure theory will actually replace set theory in mathematical practice, in particular since it does not contain any new insights into which structures or sets exist in the end, as Shapiro himself admits.[39]

8. MATHEMATICAL PREDICATES

We started our discussion of pattern structuralism with the idea that all natural number systems share something important. In order to make sense of that idea a pattern structuralist postulates a corresponding natural number pattern, as a new kind of abstract entity. What all the relational systems share is, then, that they instantiate this pattern. It might be thought that it is the ability to give this answer that should make us favor pattern structuralism over its alternatives. There is, however, a more traditional way to go here as well. Namely, we can talk about a *predicate* that applies to these and only these relational systems. In fact, this predicate was already implicit in our earlier talk about "natural number systems" – it is simply the predicate "to be a natural number system".

This basic idea can be made more formal and precise. To do so recall how we proceeded in connection with universalist structuralism. We constructed, for each arithmetic sentence p, a corresponding universal if-then sentence q of the following form:

$$\forall x \forall f \forall X[PA_2(x, f, X) \rightarrow p(x, f, X)].$$

In this sentence we used the expression:

$$PA_2(x, f, X).$$

We arrived at it from the conjunction of the Peano Axioms, in the form of $PA_2(1, s, N)$, by replacing '1' by 'x', 's' by 'f', and 'N' by 'X'. The suggestion now is this: '$PA_2(x, f, X)$' provides us with a *predicate* which applies exactly to the relational systems of which the Peano Axioms are true; and that suffices to make sense of what these systems share.

Speaking about a "predicate" here raises the following question: Are we just talking about the linguistic expression itself, as a mere formula? In other words, is this suggestion supposed to be understood in some narrowly formalist sense? Or is the formula supposed to correspond to something "behind" it, especially because of its connection with the Peano Axioms? The latter, non-formalist point of view can be spelled out in a number of different ways again. For example, we can say that the formula together with the axioms define a certain *property*; or, alternatively, that they define a certain *concept* (a second-level concept, along Fregean or Russellian lines). In either case we do appeal to something in addition to the formula itself, i.e., something '$PA_2(x, f, X)$' stands for, refers to, or expresses – again some kind of *abstract universal* or *abstract structure*. In either case we are also left with various new questions: Are the relevant concepts supposed to be identified extensionally or intensionally? Are the properties supposed to exist just in the objects which have them or also, in some sense, independently? etc.

A third alternative in this connection – one closer to current mathematical practice, especially against the background of set theory – might be the following: We can appeal to the *equivalence class* of relational structures determined by certain axioms, e.g., the class of all natural number systems. One advantage of this alternative is that it avoid any appeal to properties or concepts as additional mathematical entities. It amounts, in fact, to a reduction of the "universal" to the "particular" – the property or concept to its "extension". Then again, it leads to all the well-known problems with treating such "extensions" as particular objects. In particular, we cannot treat them simply as sets, on pain of contradiction (Russell's antinomy etc.).

If we want to avoid the "reification" of predicates altogether – either as properties or as concepts or as equivalence classes – but still not say that we are dealing with '$PA_2(x, f, X)$' as a mere formula, there is, finally, also the following route: We can maintain that what actually matters is the formula *in use*. In other words, what is meant by talking about a "predicate" in connection with it is simply that we can classify relational systems according to whether they satisfy the corresponding axioms or not. Note that this suggestion again accords quite well with common mathematical practice, especially as guided by our structuralist methodology: to formulate various sets of axioms, to see them as expressing certain "conditions" which relational systems satisfy or not, etc.[40]

We have just listed four different ways of *analyzing* our initial appeal to "predicates", and there are probably more. No matter which of these analyses we adopt – a "thicker" or "thinner" one, a more or less "reifying" one – we can note the following: This general way of looking at the situation is different from what we usually do in model-theoretic logic. There we treat '$PA_2(x, f, X)$' as an expression in a formal language. As such it is meaningless, i.e., it does not stand for anything, at least until we specify some particular interpretation for the language. Under such an interpretation some set is then assigned to it as its "meaning". But even then we are not interested in that "meaning" in itself; we are only interested in which sentences containing '$PA_2(z, f, X)$' come out "true" and which "false" under the given interpretation. In other words, model-theoretic "meaning" is just a technical tool for making this determination; it plays no further role. Here, on the other hand, we treat '$PA_2(x, f, X)$' as an *already* meaningful expression (in connection with using the Peano Axioms), in the same sense in which "to be a natural number system" is meaningful in informal mathematics. As such it is an expression with which we can make *assertions*, in particular assertions about various relational systems. That is to say, we can *use* it, namely to *predicate* something of such systems.

This last observation is what allows us to maintain the following: What mathematicians are interested in, ultimately, is to understand better such predications and predicates. In particular, mathematicians study the predicates "to be a natural number system", "to be a real number system", "to be a group", etc.[41] Of course, from a mathematical point of view a good way to study them is by identifying and comparing the relational systems to which they apply, e.g., the various natural number systems. In that respect the general appeal to "predicates" is, once more, quite compatible with methodological structuralism, indeed suggested by it.

Appealing to predicates instead of patterns may be seen to have several advantages. First, predicates are arguably more ordinary and familiar than the patterns postulated by pattern structuralism (especially if we think of predicates merely as formulas "in use"). Thus, it is clear that we can specify predicates axiomatically; mathematicians have routinely done so for quite a while, at least implicitly. Second, we can appeal to them the same way in arithmetic, analysis, and group theory; i.e., we don't need different treatments for "algebraic" and "non-algebraic" fields, or better for categorical and non-categorical theories. Third, by talking about predicates we avoid all the questions about the internal composition of patterns mentioned above; since predicates simply do not have that kind of internal composition (no matter which of our analyses above we adopt); similarly for questions about the relation of different patterns to each other, etc. In fact, from the present point of view we can see what the source of several of these questions about patterns was: a pattern structuralist tries, in a sense, to treat patterns and the positions in them *both* as universals *and* as particulars. Predicates, on the other hand, are only supposed to play the former role. The latter is delegated back to relational systems.

Then again, the appeal to predicates does not give us a full-fledged semantics and metaphysics for mathematics, the way our earlier variants of structuralism did. In other words, it does not, in itself, lead to another substantive variant of structuralism. In fact, appealing to predicates does not even, in itself, commit us to a "structuralist" view about mathematics (as, e.g., a Fregean non-structuralist appeal to concepts illustrates).[42]

9. SUMMARY AND CONCLUSION

In this paper we have identified four main variants of structuralism in the philosophy of mathematics – formalist structuralism, relativist structuralism, universalist structuralism (with modal variants), and pattern structuralism – as well as the more basic structuralist methodology that informs much of current mathematical practice. Of these the first two are seldom recognized explicitly as "structuralist" positions by contemporary philosophers, although they are quite widespread among contemporary mathematicians. The others all appear, more or less explicitly, in the recent philosophical literature, especially a modal version of universalist structuralism (Hellman, see also partly Parsons) and several slight variants of pattern structuralism (Shapiro, Resnik, partly Benacerraf and Parsons). Our main goal has been to establish, in a clear and systematic way, in which sense they are all *structuralist*, how they are *related* to each other, and what is *distinctive* about each of them.

Very briefly again, in formalist structuralism the philosophical questions which tempt us beyond the structuralist methodology itself – questions about reference and truth, about the existence and nature of "the natural numbers", "the real numbers", etc. – are given a negative, deflationary, or "thin" answer. Relativist, universalist, and pattern structuralism, on the other hand, all answer these questions in a more substantive, "thick" way. The specific answers they give are, in turn, significantly different from each other, even conflicting in many ways. In particular, both relativist and universalist structuralism are naturally understood to be eliminativist views, in sharp contrast to pattern structuralism. More basically, the semantic ideas at the core of relativist, universalist, and pattern structuralism are meant to be mutually exclusive alternatives.

If we return one more time to our initial characterization of "structuralism", the following should now be clear: All four of our variants are structuralist in the sense of conforming to intuitive thesis (1): that mathematics is primarily concerned with the "investigation of structures". But what is meant by "structure" varies considerably from case to case: it is sometimes what we have called "particular structures", i.e., relational systems or models in the ordinary mathematical sense; at other times it is "universal structures", i.e., additional abstract entities. Moreover, the latter can be understood in two different ways again: as "patterns" or as "predicates". Relatedly, several of our variants agree with intuitive thesis (2): that doing mathematics involves "an abstraction from the nature of individual objects". However, how the corresponding "abstraction" is to be thought of differs considerably between them. According to one variant, (2) is even replaced by (3): that individual mathematical objects "have no more to them than can be expressed in terms of the basic relations of the structure".

Two final remarks: First, the purpose of this paper was *not* to argue in favor of one of our four variants of structuralism over against the others; nor was it to argue for or against structuralism in general. Rather, it was to show, in a more preliminary manner, that these variants are *significantly different* from, as well as *interestingly related* to, each other. The difficulties raised in connection with each of them were, thus, meant to highlight these differences and relations, not to decisively refute any of the positions. Second, while we think that we have covered all the main structuralist positions in the recent philosophical literature, our list of variants is *not* meant to be necessarily *complete*. In fact, we would claim that the position of Richard Dedekind – the thinker often appealed to by contemporary structuralists as their distinguished forefather – represents a noteworthy additional variant of structuralism. Moreover, Dedekind's structuralist position is in some ways more attractive than any of the currently prominent ones. To defend these last two claims will, however, require another paper.

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NOTES

[1] For references see the bibliography. Similar structuralist themes can also be found in Steiner (1975) and Jubien (1977), among others. For historical sources see Dedekind (1872, 1887) (we plan to come back to an investigation of his views in another paper; compare also Tait (1996)), as well as the references to Bourbaki, Hilbert, Poincaré, Quine, and Russell later in this paper.
[2] Sometimes they are acknowledged in passing; see, e.g., Hellman (1996, 100): "As with many 'isms', 'structuralism' is rooted in some intuitive views or theses which are capable of being explicated and developed in a variety of distinct and apparently conflicting ways". After this opening remark Hellman goes on to focus exclusively on his own variant. The most extended discussion of the corresponding differences can be found in Shapiro (1997, chs. 3–7). Our discussion in this paper (conceived largely before the appearance of this book) will overlap with Shapiro's to some degree. Our conception of structuralism will, however, be both broader and more fine-grained than his; i.e., we will discuss some variants of structuralism not considered by him, add further historical references, and focus more on semantic issues.
[3] Two of the most interesting and influential discussions of structuralism in general, Benacerraf (1965) and Parsons (1990), are, in our view, examples of such blurring of distinctions (for details see later). In fact, it was puzzlement about what Parsons means by "the structuralist view" that led the two present authors to work on this issue further. (Parsons has since clarified his views, e.g., in Parsons (1997).)
[4] One can, of course, also use the language of 1st-order logic plus set theory to formulate the Peano Axioms (especially (A3), the induction axiom); see, e.g., Parsons (1990, 306), and Gleason (1991, 89). But for our purposes this approach is less appropriate, since we do not want to assume that the Peano Axioms are intrinsically set-theoretic (for reasons that will become clear later).
[5] For an explicit formulation of the axioms for a complete ordered field see, e.g., Mac Lane (1986, 102–124), or Gleason (1991, 94–111); for the group axioms see Mac Lane (1986, 23–26).
[6] What we call "natural number systems" have also been called "simply infinite systems" (Dedekind (1887)), "progressions" (Russell, Quine, etc., see the corresponding quotes later on), and "ω-sequences" (Zermelo etc.). What we call "relational systems" are often called "relational structures" or simply "structures" in the literature, especially when set theory is taken as part of the background (see, e.g., Enderton (1977, 170), and van Dalen (1994, 56). As we will later want to distinguish between several different kinds of "structure" (in Sections 6, 7, and 8), we use the more neutral "system" here (appropriated from Dedekind (1887); compare also Shapiro (1997)).
[7] A set S is Dedekind-infinite if and only if there exists a one-to-one function f that maps S into a proper subset S' of itself (Dedekind (1887), Definition 64). For the results that follow in this section of the paper compare Dedekind (1887), Enderton (1977, ch. 4), and Gleason (1991, ch. 7.)
[8] For an interesting historical and philosophical discussion of how several of these developments arose in the 19th century see Stein (1988). For more on how they progressed into the 20th century and how they relate to "structuralism" see Corry (1996), Dieudonné (1979), Mac Lane (1996), and Shapiro (1997, ch. 5).
[9] More recently an alternative framework – still within the general confines of our structuralist methodology – has been provided by category theory. We will not try to explore the consequences of a shift from set theory to category theory in this paper. See Mac Lane (1971, 1986, 1996), McLarty (1992, 1993), and Awodey (1996) for explicit categorical perspectives on structures in mathematics and logic.

[10] As a structuralist approach typically involves the use of full classical logic (infinite sets, undecidable predicates, non-constructive existence proofs, etc.), constructivist and intuitionist mathematicians have often resisted it, or at least its full development. An early, classical example of such resistance is that of Kronecker to the work of Cantor and others, as mentioned in Stein (1988).
[11] For more on Bourbaki in this connection see again Dieudonné (1979), also Corry (1992, 1996).
[12] Sometimes the epistemological implications of a structuralist approach are also at issue; see Resnik (1982, 1997, chs. 6–9), and Shapiro (1997, ch. 4). For the sake of brevity we suppress them here. – We are, we would like to emphasize, not claiming that such semantic, metaphysical, and epistemological questions exhaust the philosophy of mathematics, nor that they are the most interesting questions from a mathematical point of view. Their discussion has, however, dominated large parts of recent philosophy of mathematics, thus inviting a reflection on what is really at issue in them.
[13] We have encountered this kind of response repeatedly in conversations with mathematicians, especially when pressed on corresponding philosophical issues. David Hilbert's position is also often interpreted along these lines (although as we understand him Hilbert's formalism is really methodological, i.e., meant to justify various parts of mathematics that are not completely meaningless in the end). Finally, compare the (somewhat cryptic) formalist suggestions at the very end of Benacerraf (1965).
[14] We are here close to what is sometimes called "if-then-ism" in the literature. For references and further discussions see Putnam (1967) (whose suggestions, including his "modal turn", were later transformed into an explicit form of "universalist structuralism" by Geoffrey Hellman; cf. Section 5 below) and Rheinwald (1984, ch. 2) (where if-then-ism is discussed under the general rubric of "formalism").
[15] In our usage to qualify as a "variant of structuralism", in the philosophical sense, involves not just adopting the structuralist methodology, but also giving answers to our semantic and metaphysical questions, more specifically answers that do not include essentially anti-structuralist elements. The three versions of formalist structuralism considered so far are, then, variants of structuralism, but only in a minimal or marginal sense. We will identify three more central or substantive variants in what follows: what we will call "relativist", "universalist", and "pattern structuralism", respectively.
[16] The qualification "in the language of arithmetic" is important. Consider, e.g., the finite Zermelo ordinals and the finite von Neumann ordinals as forming two models of arithmetic. Then there are "non-arithmetic" statements, such as "$0 \in 2$" or "$1 \subseteq 2$", that are false in the one, but true in the other model (similarly the other way around). But these differences do not matter with respect to "truth in arithmetic".
[17] See here (Hodges, 1986, 148-150). Hodges compares the use of non-logical constants explicitly to the use of 'he', 'yesterday', and 'the latter' in ordinary English.
[18] While much of the recent focus on one's "ontological commitments" in mathematics, including the demand to be economic or parsimonious about them, seem to have their source in Quine, the appeal to "ontological economy", "Occam's Razor", etc. actually goes back at least as far as Russell (as does the discussion of several other issues with which we will be concerned later in this paper); see Russell (1903, 72), also (1919, 184). Then again,

the claim that economy with respect to one's mathematical ontology is a necessary, or even a commendable, attitude is not universally shared; see Burgess (1998).

[19] See Goodman and Quine (1947), Field (1980), partly also the corresponding discussion in Parsons (1990).

[20] For a comprehensive discussion of mereology and related issues see, e.g., Simons (1987).

[21] A few pages later Benacerraf goes even further: "I therefore argue, extending the argument that led to the conclusion that numbers could not be sets, that numbers could not be objects at all; for there is no more reason to identify any individual number with any one particular object than with any other (not already known to be a number)" (Benacerraf 1965, 290–291).

[22] A less misleading formulation, suggested to us by an anonymous referee, would be this: there are no *natural* natural numbers. – Note that the argument just presented presupposes that all relevant models of PA_2 are constructed within set theory or some analogous theory, i.e., are "homogeneous" in a crucial way. Without this presupposition, in particular if we allow the natural numbers to be *sui generis*, the argument looses its force (in spite of what Benacerraf writes elsewhere; compare the previous footnote.)

[23] It is because of this third step, in particular because we want to quantify out 'N', that we have formulated the Dedekind–Peano Axioms differently in this section. (We could, of course, have used this second formulation already earlier; but that would have complicated the semantics in Section 4 slightly.)

[24] In this section our discussion is mostly guided by Hellman (1989, 1990a, 1990b, 1996), to some degree also by Parsons (1990). Both of them make explicit use of universal if-then sentences similar to ours. However, with respect to Parsons compare footnotes 3, 26, and 31; and with respect to Hellman we will have to come back to his modal "twist" later in this section. Finally, for historical roots of this kind of view compare our quotations from Russell later on in this section.

[25] This aspect of universalist structuralism is seldom discussed in any depth in the current literature, in spite of the fact that Russell was already explicit about it. Thus in Russell's *Principles of Mathematics* it comes up early, as follows: "The above definition of pure mathematics [as consisting of universal if-then sentences, see above] [...] professes to be, not an arbitrary decision to use a common word in an uncommon signification, but rather a *precise analysis* of the ideas which, more or less unconsciously, are *implied* in the ordinary employment of the term" (Russell 1903, 3, our emphasis). (In this passage Russell talks primarily about analyzing the term 'pure mathematics'; but this commits him also to analogous views about analyzing individual mathematical propositions.)

[26] To be precise here, the eliminativism aimed for in universalist structuralism may be only partial, e.g., if we allow for sets, but want to eliminate all other abstract objects (see below). On the other hand, compare Hellman who states quite strongly: "[W]e seek an alternative, non-literal interpretation of mathematical discourse [...] in which ordinary quantification over abstract objects is *eliminated entirely*" (Hellman 1989, 2, our emphasis; more on Hellman below). In Parsons (1990) both full-blooded and partial eliminativist goals connected with universalist structuralism are discussed as well, although Parsons himself endorses neither of them in the end.

[27] See Hellman (1989, 1990a, 1990b, 1996). Note, in particular, his use of the modal calculus *S5* to explain away the assumption of *possibilia*. For a different appeal to modality in a similar context compare Kalderon (1996). Both Hellman and Kalderon present themselves as influenced by Putnam (1967).

[28] Note that set theory can be seen as providing a sufficient, perhaps also a necessary, condition for the possibility involved here: in the form of satisfiability in the set-theoretic universe. A thoroughgoing modal structuralist will, however, avoid relying on set theory that way; see Hellman (1990b). Compare in this connection also the discussion of "coherence" in Shapiro (1997, chs. 4 and 6), with explicit references to Hellman's "logical" modalities. (We will come back to the same problem in connection with Shapiro's own views later.)

[29] Compare here Putnam (1967), Hellman (1989, chs. 1–2), Kalderon (1996), as well as the related discussions in Resnik (1997, ch. 4), and Shapiro (1997, chs. 6–7). Note that for Hellman the attraction of a modal-structuralist approach seems to come more from underlying metaphysical considerations, in particular the felt need to do without abstract objects, than from strong claims about what is implicit in mathematical practice (see Hellman 1989, Introduction). In addition, there is no indication that he wants to change ordinary mathematical practice, say arithmetic, by somehow introducing modal considerations into it.

[30] In Parsons (1990) several such attempts are discussed in detail.

[31] Note how different this characterization is from the one implicit in our original quote from Parsons: "[M]athematical objects [...] have no more to them than can be expressed in terms of the basic relations of the structure" (*ibid.*, 303). That quote suggests the idea of a "pattern" (see Section 7).

[32] For other examples of this use of "structure" in the current mathematical and logical literature see, e.g., Gleason (1991, 55), and compare the references in the second half of fn. 6. For historical sources see Bourbaki (1950), Dieudonné (1979), and Corry (1996). Finally, see Bourbaki (1954–56, ch. IV), where a related, precise, but forbiddingly complex general definition of "mathematical structure" is given.

[33] Michael Dummett, also observing the ambiguous use of "structure" in the literature, has recently proposed a similar distinction. As he puts it: "There is an unfortunate ambiguity in the standard use of the word 'structure', which is often applied to an algebraic or relational system – a set with certain operations or relations defined on it, perhaps with some distinguished elements; that is to say, to a model considered independently of any theory which it satisfies. This terminology hinders a more abstract use of the word 'structure': if, instead, we use 'system' for the foregoing purpose, we may speak of two systems as having an identical structure, in this more abstract sense, just in case they are isomorphic. The dictum that mathematics is the study of structures is ambiguous between these two senses of 'structure' " (Dummett 1991, 295). Note, however, that, while our distinction between "particular" and "universal" structures corresponds closely to his between "concrete" and "abstract" ones, Dummett's use of "abstract" in this connection is potentially misleading (as is Benacerraf's, see (1965, 291) etc.). After all, a set-theoretic relational system such as the finite von Neumann ordinals is already "abstract" in some sense, isn't it? Thus a further distinction between degrees or kinds of "abstractness" would be needed.

[34] Note however, the following argument: Sequences of physical objects, temporal moments, etc. do *not* exemplify the natural number pattern *strictly*; or better, they do so only if we conceive of them in an *idealized* form. This point, derived from Plato, is emphasized in Tait (1986a, 1986b). (It also constitutes a further difficulty for nominalist versions of relativist and universalist structuralism; see Sections 4 and 5.)

[35] Michael Resnik, in whose early writings on the topic a similar kind of realism was proposed, has recently been more reserved. Thus with respect to "positing an ontology of featureless objects, called 'positions', and construing structures as systems of relations or

'patterns' in which these positions figure" he now wants to restrict himself to a "methodology of ontology" of mathematics, or to an "epistemic reading" (Resnik 1997, 269). What exactly that means is, however, not immediately clear; for further elaboration see Resnik (1997) (and compare fn. 38).

36 Resnik himself writes elsewhere: "Group theory does not posit a subject matter; when it needs examples of groups it turns to other branches of mathematics" (Resnik 1997, 264). He also acknowledges: "[I]t is more plausible that 2nd-order set theory is concerned with a unique structure than that 1st-order set theory is" (*ibid.*, 263).

37 As an anonymous referee has reminded us, Shapiro's distinction between "algebraic" and "non-algebraic theories" is not altogether happy. After all, there are various mathematical theories outside of algebra that have many non-isomorphic models, e.g., that of Riemannian manifolds, of linear topological spaces, etc.; conversely, a number of algebraic theories have only one model up to isomorphism, e.g., the theory of abelian groups of exactly five elements. It is better, then, to make the distinction in terms of "categorical" vs. "non-categorical theories" (as Shapiro himself does elsewhere implicitly).

38 Resnik acknowledges this as a problems when he writes: "[I do not] see how to express the idea that a pattern 'consists' of a domain of positions and relations on that domain" (Resnik 1997, 256). Actually, in the end Resnik treats the appeal to patterns, positions, etc. as less than a full-fledged theoretical position anyway. He writes: "I have neither the talent nor the taste for carrying out the project [of constructing a pattern theory]. Moreover, I see no current mathematical need for doing so" (*ibid.*, 257) This leads him, among others, to be rather "cautious about asserting the existence of patterns" (*ibid.*, 261; cf. footnote 35). Shapiro, in contrast, explicitly proposes a systematic pattern theory (see below).

39 Shapiro puts this last issue in terms of questions about "coherence"; see Shapiro (1997, 132–136) etc. His explicit discussions of it, including various remarks about the related notions of "consistency", "satisfiability", etc., are among the most interesting aspects of his book. Still, their upshot is, as he himself makes clear, that his structure theory gives *no more nor less* of an answer to such existence questions than either set theory or Hellman's modal structuralism.

40 The proposal in the last paragraph is implicitly guided by a certain interpretation of Dedekind's work. Thus we see him as one of the first mathematicians to advocate such a point of view, e.g., concerning the Dedekind–Peano Axioms in Dedekind (1887). (We plan to come back to Dedekind's views in a separate paper; compare also fn. 1 and the remark at the very end of this paper.)

41 Another example is the predicate "to be a Euclidean Space" as determined by the list of axioms given in Hilbert (1899). Note that, on a charitable interpretation, this is exactly what Frege in his well-known debate with Hilbert took Hilbert's axioms to do: to specify a higher-order predicate (for him: concept); see Frege (1906). Frege himself seems, however, not to have been entirely clear on this point, as the corresponding meta-theoretic point of view was far from his usual way of thinking.

42 Predicates in our sense have not found much attention in recent philosophy of mathematics (but see the previous two footnotes for more historical references). On the other hand, in contemporary philosophy of science they have played some role, and expressly in connection with a "structuralist program"; see Sneed (1971), more recently also Balzer, Moulines and Sneed (1987). Note, in particular, remarks such as the following: "The essence of this [structuralist] doctrine is that all such informal axiomatizations of mathematical theories may be regarded as, more or less, adequate definitions of set-theoretic *predicates* – that is roughly, *predicates* definable with the conceptual apparatus of set theory. (Sneed 1971, 9, our emphasis). Note also that Sneed's and his followers' approach is partly based on earlier work done by Patrick Suppes, e.g., in Suppes (1957). In Suppes' work theories from mathematical physics and from pure mathematics are treated correspondingly.

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